Polynomial Interpolation & Numerical Integration

Goal:- let $a<b$ and $a, b \in \mathbb{R}$
· $f:[a,b] \to \mathbb{R}$

· Find $P(\cdot) \equiv P_n(\cdot)$ - Polynomial of degree 'n' that "approximates" f

Applications:-

· use $P(\cdot)$ to approximate f within the range of data points

· Find $\int_a^b f(x)\,dx$

Accuracy:-

· Find Error bounds in approximation
· Computational Efficiency
· width of $[a,b]$ - Composite methods

Connections:- - best linear fit [least-square line]
- Trignometric interpolation.

Polynomial Interpolation

let (x_i, y_i) $0 \leq i \leq n$ be a set of $n+1$ points on the plane, with $x_i \neq x_j$, $i \neq j$.

Definition :- A polynomial $p: \mathbb{R} \to \mathbb{R}$ is said to interpolate $\{(x_i, y_i) \mid 0 \leq i \leq n\}$ if

$$p(x_i) = y_i \quad \text{for} \quad 0 \leq i \leq n.$$

The points x_i are refered to as nodes.

Example :. $n = 1$ (x_0, y_0) & (x_1, y_1) are two points

$$p(x) = y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$$

is a 1-degree polynomial that interpolates the two points. This is known as linear interpolation.

One can rewrite $p(\cdot)$ in the following manner

$$p(x) = \left(\frac{x - x_0}{x_1 - x_0}\right) \cdot y_0 + \left(\frac{x - x_1}{x_1 - x_0}\right) y_1$$

This representation generalises as we shall see below.

Theorem:- let (x_i, y_i) $0 \le i \le n$ be a set of $n+1$ distinct points on the plane. Then there exists a unique polynomial $p: \mathbb{R} \to \mathbb{R}$ of degree at most n such that

$$p(x_i) = y_i \qquad 0 \le i \le n.$$

Proof:- Existence :- By induction

$n=1$: Example

Assume $n=k$

$n = k+1$ $\{(x_i, y_i) \mid 0 \le i \le k+1\}$

let $p: \mathbb{R} \to \mathbb{R}$ be a polynomial of degree at most k that interpolates $\{(x_i, y_i) \mid 0 \le i \le k\}$.

$g: \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = p(x) - c \prod_{i=0}^{k} (x - x_i) \qquad \text{for some } c > 0$$

$$g(x_i) = y_i \qquad 0 \le i \le k$$

let $c = \dfrac{p(x_{k+1}) - y_{k+1}}{\prod_{i=0}^{k} (x - x_i)}$

$$\Rightarrow g(x_{k+1}) = y_{k+1}$$

$\Rightarrow$ g is a polynomial of degree at most $k+1$ and interpolate $\{(x_i, y_i) \mid 0 \le i \le k+1\}$.

Uniqueness :- Exercise $\square$

Question :- How does one find a polynomial given $\{(x_i, y_i) : 0 \le i \le n\}$ points ?

I. Use Monomials :-

let $p : \mathbb{R} \to \mathbb{R}$

$$p(x) = \sum_{k=0}^{n} a_k x^k \qquad a_k - \text{unknown}$$

$p(x_i) = y_i$

$$\Rightarrow \quad Va = y$$

where

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & & & & \\ \vdots & & & & \\ 1 & x_n & x_n & \cdots & x_n^n \end{bmatrix} \qquad y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

$$a = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$$

One could try to invert - Vandermonde matrix V to solve for a. We will see later that this is a hard problem computationally. Example :- x_i is close to x_j $i \ne j$

observation :- If we let

$$\mathbb{P}_n[a,b] = \left\{ p : [a,b] \to \mathbb{R} \;\middle|\; \begin{array}{l} p \text{ is a polynomial of} \\ \text{degree at most } n \\ \text{with real coefficients} \end{array} \right\}$$

$\mathbb{P}_n[a,b]$ is a vector space &

$\{1, x, \ldots x^n\}$ are the monomial basis.

II Lagrange Polynomials :-

let $n \geq 1$ be fixed. and
$(x_0, \dots, x_n)$ be $n+1$ distinct points

Define :- The Lagrange polynomials for the interpolation points $\{x_0, \dots x_n\}$ are given by

$$l_i(x) \equiv l_i^{(n)}(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - x_k)}{x_i - x_k}$$

for $0 \leq i \leq n$.

Note :- For each i, $0 \leq i \leq n$, $l_i(\cdot)$ are polynomials of degree n. and

$$l_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & i = j \end{cases}$$

let $\{(x_i, y_i) : 0 \leq i \leq n\}$ be given. Then $p: \mathbb{R} \to \mathbb{R}$

$\therefore$ $$p(x) = \sum_{i=0}^{n} y_i \, l_i(x)$$ is a polynomial that interpolates the points.

Observe :- Seen earlier in $n=1$ case

$$p(x) = y_0 \frac{(x - x_0)}{x_1 - x_0} + y_1 \frac{(x - x_1)}{x_1 - x_0}$$

Numerical Integration :

Question :-

$f : [a,b] \to \mathbb{R}$

$$\int_a^b f(x)\,dx = ?$$

[Riemann integral]

Approaches :-

- $F : [a,b] \to \mathbb{R}$ and $F'(x) = f(x)$

  then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- Riemann Sums :- f - continuous.

$$a \le x_0 \le \dots \le x_n \le b \qquad x_i = a + \frac{i(b-a)}{n}$$

$$R_n(f) := \frac{1}{n} \sum_{i=1}^{n} f(x_i) \longrightarrow \int_a^b f(x)\,dx$$

as $n \to \infty$

- Need to understand rate at which

$$\alpha_n = \left| R_n(f) - \int_a^b f(x)\,dx \right|$$

goes to zero.

- other approximations.

Examples :- $\displaystyle\int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = ?$

require good approximating procedures, with error bounds

- $p: \mathbb{R} \to \mathbb{R}$ - polynomial of degree n then

$$p(x) = \sum_{k=0}^{n} c_k x^k$$

$$\int_a^b p(x)\, dx = \sum_{k=0}^{n} \frac{c_k}{k+1} \left(b^{k+1} - a^{k+1} \right)$$

Quadrature / Numerical Integration methods :-

$a < b$ - fixed.

Take - points in $[a,b]$ say $\{ x_0, \ldots, x_n \} = X_n$

Find p: polynomial that interpolates

$$\{ (x_i, f(x_i)) \mid 0 \le i \le n \}$$

Compute $\displaystyle\int_a^b p(t)\, dt$

Error $\displaystyle\int_a^b f(x)\, dx - \int_a^b p(t)\, dt$

<u>Lagrange Polynomial</u> that interpolates is given by

$$p(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

$$\int_a^b p(x)\,dx = \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x)\,dx$$

let $w_i = \int_a^b l_i(x)\,dx$

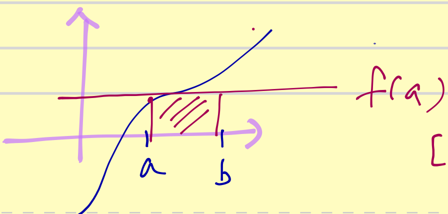
Then approximation for

$$\int_a^b f(x)\,dx \approxeq \sum_{i=0}^{n} f(x_i)\, w_i$$

Cases :-

(i) $X = \{a\}$

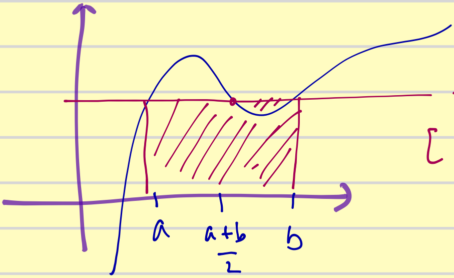
$p(t) = f(a)$ $\qquad \int_a^b p(t)\,dt = f(a)(b-a)$



[Rectangle Rule]

(ii) $X = \left\{ \frac{a+b}{2} \right\}$

$p(t) = f\left(\frac{a+b}{2}\right)$ and $\int_a^b p(t)\,dt = f\left(\frac{a+b}{2}\right)(b-a)$

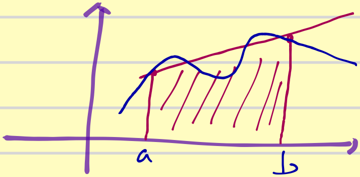


$f\left(\frac{a+b}{2}\right)$

[midpoint Rule]

(iii) $X = \{a, b\}$

$p(t) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$

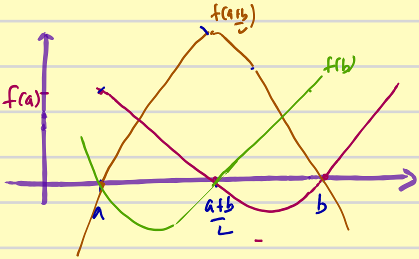


[Trapezoid Rule]

$\int_a^b p(t)\,dt = \cdots = [f(b) + f(a)] \frac{(b-a)}{2}$

(iv) $X = \{ a, \frac{a+b}{2}, b \}$

$$p(t) = \frac{(t-b)\left(t-\left(\frac{a+b}{2}\right)\right)}{(a-b)\left(a-\left(\frac{a+b}{2}\right)\right)} f(a) + \frac{(t-a)(t-b)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} f\left(\frac{a+b}{2}\right) + \frac{(t-a)\left(t-\frac{a+b}{2}\right)}{(b-a)\left(b-\frac{a+b}{2}\right)} f(b)$$



$$\int_a^b p(t)\,dt = f(a) \int_a^b \frac{(t-b)\left(t-\left(\frac{a+b}{2}\right)\right)}{(a-b)\left(a-\left(\frac{a+b}{2}\right)\right)}\,dt + f\left(\frac{a+b}{2}\right) \int_a^b \frac{(t-a)(t-b)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\,dt$$

$$+ f(b) \int_a^b \frac{(t-a)\left(t-\frac{a+b}{2}\right)}{(b-a)\left(b-\frac{a+b}{2}\right)}\,dt$$

$$= \underbrace{\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]}_{\text{Simpson's Rule}}$$

Question:- Can we quantify the error?

Error Analysis [Lagrange Polynomials]

let $P(\cdot)$ be the Lagrange Polynomial which interpolate $\{(x_i, f(x_i)) : 0 \le i \le n\}$

$$P(x_i) = f(x_i) \quad \text{if } 0 \le i \le n$$

Fix, $x \neq x_i$ $\quad g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \left(\frac{t - x_i}{x - x_i} \right)$$

Now $\quad g(x_i) = 0 \quad 0 \le i \le n$ $\qquad$ Assume: $f \in C^{n+1}[a,b]$

& $g(x) = 0$

- g has $n+2$ zeros

[Inductively] Rolle's Theorem $\Rightarrow \exists\, \xi(x)$ between $x, x_0, x_1, \ldots, x_n$ such that $\quad g^{n+1}(\xi) = 0$

- $P^{n+1}(\xi) = 0$

- Observe $$\prod_{i=0}^{n} \frac{t - x_i}{x - x_i} = t^{n+1} \frac{1}{\prod_{i=0}^{n} (x - x_i)} + h(t)$$

with $h(t)$ = polynomial in t of degree less than $\le n$.

$$\Rightarrow \frac{d^{n+1}}{dt^{n+1}}\left(\prod_{i=0}^{n}\frac{t-x_i}{x-x_i}\right) = \frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

$\therefore \quad g^{n+1}(\xi) = 0 \quad \Rightarrow$

$$\Rightarrow \quad 0 = f^{n+1}(\xi) - P^{n+1}(\xi) - [f(x) - P(x)] \left.\frac{d^{n+1}}{dt^{n+1}}\prod_{i=0}^{n}\left(\frac{t-x_i}{x-x_i}\right)\right|_{t=\xi}$$

$$\Rightarrow \quad 0 = f^{n+1}(\xi) - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

$$\Rightarrow \quad f(x) = P(x) + \frac{f^{n+1}(\xi)\prod_{i=0}^{n}(x-x_i)}{(n+1)!} \quad - (*)$$

$(*)$ holds trivially at $x = x_i \quad 0 \leq i \leq n$.

Theorem 1 :- $f : [a,b] \to \mathbb{R}$ f be $(n+1)$ times differentiable.
Let $\{x_i : 0 \leq i \leq n\}$ be $n+1$ - distinct points in $[a,b]$
Let $p(\cdot)$ be the Lagrange Polynomial interpolating $\{(x_i, f(x_i)) : 0 \leq i \leq n\}$.
Then $\forall x \in [a,b]$ $\xi(x)$ between $x_0, \ldots x_n$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))\prod_{i=0}^{n}(x-x_i)}{(n+1)!}$$

Error analysis for integrals using Lagrange Polynomials

Apply Theorem 1 to get:

$$\int_a^b f(x)\,dx = \int_a^b p(t)\,dt + \int_a^b \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x-x_i)\,dx.$$

Rectangle rule :-

continuous

$$\text{Error}_{\text{Rectangle}} = \int_a^b f'(\xi(x))\,\underbrace{(x-a)}_{\geq 0}\,dx = f'(c)\int_a^b (x-a)\,dx$$

(MVT for integrals)

$$= \frac{f'(c)}{2}(b-a)^2, \text{ for some } c\in(a,b).$$

Trapezoid rule :-

continuous

$$\text{Error}_{\text{Trapezoid}} := \int_a^b f''(\xi(x))\,\underbrace{(x-a)(x-b)}_{\leq 0}\,dx$$

$$= \frac{f''(c)}{2}\int_a^b (x-a)(x-b)\,dx$$

(MVT for integrals) $$= -\frac{f''(c)\,(b-a)^3}{12}, \text{ for some } c\in(a,b).$$

Midpoint rule

$$\text{Error}_{\text{Midpoint}} = \int_a^b f'(\xi(x))\, e(x)\, dx$$

with $\quad e(x) = x - \frac{(a+b)}{2}$

rule

$$\text{Error}_{\text{Simpson}} = \int_a^b \frac{f'''(\xi(x))}{6}\, e(x)\, dx$$

$$e(x) = (x-a)\left(x - \frac{(a+b)}{2}\right)(x-b)$$

In both of these cases we note that $e(\cdot)$

- Does not have same sign $\quad$ and $\quad \int_a^b e(x)\,dx = 0$.

The error analysis is a bit different.