Newton-Cotes Quadrature :- $f : [a,b] \to \mathbb{R}$

$x_i \in [a,b]$ $\quad x_i \neq x_j$ $\quad 0 \le i,j \le n.$

$P_n(\cdot)$ – interpolating polynomial

$$\int_a^b f(x)\,dx = \int_a^b P_n(x) + E_n(f)$$

with
$$E_n(f) = \int_a^b \frac{f^{n+1}(\xi(x))}{(n+1)!} \prod_{j=0}^{n} (x-x_j)\,dx$$

Trapezoid rule: $\{x_0, x_1\} = \{a, b\}$

- Precision upto polynomials of degree 1.

Simpson rule :- $\{x_0, x_1, x_2\} = \{a, \frac{a+b}{2}, b\}$

- Precision upto polynomial of degree 3.

Question :- Can we do better ?

Answer :- Gauss-Legendre Quadrature

We observed that if $\{x_0, \ldots, x_n\}$ as nodes $P_n(\cdot)$ was the interpolating polynomial

- $f(x) = P_n(x)\,dx + f[x_0, \ldots x_n, x] \prod_{i=0}^{n} (x - x_i)$ and

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^b P_n(x)\,dx + E(f) \quad \text{where}$$

$$E(f) = \int_a^b f[x_0, \ldots, x_n, x] \prod_{j=0}^{n} (x - x_j)\,dx$$

- Let $f : [a,b] \to \mathbb{R}$ be a polynomial of degree $n+m$; $\{x_i : 0 \le i \le n\} \subseteq [a,b]$ $x \in [a,b]$

Fact :- $f[x_0, \ldots x_n, x] = \sum_{k=0}^{m} \alpha_k x^k$, for some $\alpha_k \in \mathbb{R}$

$$E(f) = \int_a^b f[x_0, \ldots, x_n, x] \underbrace{\prod_{j=0}^{n} (x - x_j)}_{\psi(x)}\,dx$$

$$= \sum_{k=0}^{m} \alpha_k \int_a^b x^k \psi(x)\,dx$$

- KEY IDEA :- Choose $q : [a,b] \to \mathbb{R}$, polynomial with zeros $\{x_0, x_1, \ldots, x_m\}$

$$\& \quad \int_a^b x^k q(x)\,dx = 0 \qquad k = 0, 1, \ldots, m$$

Methodology:- Take $x_0, x_1, \ldots, x_n$ to be the zeros of a polynomial $q(\cdot)$ of degree $n+1$ such that

$$\int_a^b x^k q(x)\,dx = 0 \qquad k = 0, 1, 2, \ldots, n.$$

Take: $P_n(\cdot)$ to be interpolating polynomial on $\{(x_i, f(x_i)) : 0 \le i \le n\}$

Then
$$\int_a^b f(x)\,dx = \int_a^b P_n(x)\,dx + E(f)$$

where
$$E(f) = \int_a^b f[x_0, \ldots x_n, x] \prod_{i=0}^{n} (x - x_i)\,dx$$

and $E(f) = 0$ if f is a polynomial of degree at most $2n+1$

Some Prescriptions:- $\int_{-1}^{1} f(x)\,dx$

Legendre - Polynomial : $q_n(1) = 1$

$$q_0(x) = 1, \quad q_1(x) = x$$

$$q_{n-2}(x) = \left(\frac{2n-1}{n}\right) x\, q_{n-1}(x) - \left(\frac{n-1}{n}\right) q_{n-2}(x)$$

$$\left[\, q_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad q_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \,\right]$$

- There is a table of nodes & weights available upto $n=95$

- R - has a package to do Gauss-legendre quadrature.

- Also $\int_a^b f(x)\,dx$ is possible

by a change of variable:

Set: $u = \frac{b-a}{2}(x-a) - 1$

$$x = \frac{2u+2}{b-a} + a$$

So $\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{2u+2}{b-a} + a\right) du$

and we can always work with $[-1,1]$

- Also use Composite Gauss-Legendre

& Adaptive Gauss-Legendre

Can use old error bounds

have precision upto $2n+1$

$$E_n(f) = \int_a^b \frac{f^{n+1}(\xi(x))}{(n+1)!} \prod_{j=0}^{n} (x-x_j)\,dx$$

Things to do during break :-

- Catch up on material - Solve midterms Problems

- Extra Credit Problems - CS II
  - One submission (Nitya)
  - Please submit anytime through the two weeks

- Homework 9 - CS II
  - due Thursday 19^{th} March
  - Slip into my office door
  - upload pdf on dropbox folder (latex / hand write + photo)

- Online class :
  - next Thursday 10:00 am

- Feed back Survey :-
  - please fill it out as soon as you can

- will post recording of class & notes on website.