

Polynomial Interpolation & Numerical Integration

$$a, b \in \mathbb{R} \quad a < b$$

Goal:- $f: [a, b] \rightarrow \mathbb{R}$

Find $n \geq 1$, $P_n(\cdot)$ a polynomial of degree at most n ($P_n(\cdot)$) that "approximates" f from information at $\{(x_i, f(x_i)) \mid 0 \leq i \leq n\}$
 $x_i \in [a, b] \quad (i = 0, \dots, n)$

Applications:

- approximate f with P "inside" the range of the data points
- Find $\int_a^b f(x) dx$ with help of $\int_a^b P(x) dx$

Accuracy:

- Error bounds in approximation
- Computational efficiency. $\underbrace{b-a}_{\text{width of } [a,b]}$ - composite methods

Connections:

[not necessarily interpolation]

- Taylor Polynomial
- Given 'n' data points - Best line fit [least square line]
- Trigonometric approximation

Polynomial Interpolation

Let $\{(x_i, y_i) : 0 \leq i \leq n\}$ be points in $\mathbb{R}^2$

Such that $x_i \neq x_j \quad \forall i \neq j$

Definition — A polynomial $p: \mathbb{R} \rightarrow \mathbb{R}$ is said to interpolate $\{(x_i, y_i) : 0 \leq i \leq n\}$ if

$$p(x_i) = y_i \quad \forall i \in \{0, 1, \dots, n\}$$

The points are referred to as nodes.

Example

$n=1$

$\{(x_0, y_0), (x_1, y_1)\}$

$x_0 \neq x_1$

$$p(x) = y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0} \right) (x - x_0)$$

is a polynomial of degree 1 ; $p(x_i) = y_i$

This is linear interpolation

It is convenient to rewrite

$$\begin{aligned} p(x) &= y_0 \left(\frac{x - x_1}{x_0 - x_1} \right) + y_1 \left(\frac{x - x_0}{x_1 - x_0} \right) \\ &= y_0 l_0(x) + y_1 l_1(x) \end{aligned}$$

Theorem:

Let $\{(x_i, y_i) : 0 \leq i \leq n\}$ be set of $n+1$ points in $\mathbb{R}^2$ such that $x_i \neq x_j$ for $i \neq j$. Then there exists a unique polynomial of degree at most n , $p: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$p(x_i) = y_i \quad i = 0, \dots, n.$$

Proof:-

Existence:- By induction on n

Uniqueness: Fundamental Theorem of Algebra. $\square$

Question - Given $\{(x_i, y_i) \mid 0 \leq i \leq n\}$ how does one find the $P: \mathbb{R} \rightarrow \mathbb{R}$ polynomial of degree at most n such that $P(x_i) = y_i$ for $0 \leq i \leq n$?

Answer I - monomial basis $= \{1, x, \dots, x^n\}$

let $P: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$P(x) = \sum_{k=0}^n a_k x^k \quad \text{with } a_k \in \mathbb{R}$$

To find $\{a_k\}_{k=0}^n$ solve

$$p(x_i) = y_i$$

$\Rightarrow$ solve

$$Va = y$$

$$a = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$$

$$y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

If $\lambda_i \neq \lambda_j$ for $i \neq j$ then it is well known

$Va = y$ has a unique solution

given by $a = V^{-1}y$

Issues:

n - large and $\lambda_i \in [a, b]$ $0 \leq i \leq n$

$\Rightarrow \lambda_i$'s are close together

$\Rightarrow$ determinant of V is close to 0

(discuss) - V is ill conditioned matrix
later in course

Lagrange Polynomials

let $n \geq 1$ and $\{(x_i, y_i) \mid 0 \leq i \leq n\}$ be $n+1$ data points in the plane $\mathbb{R}^2$ $x_i \neq x_j$ for $i \neq j$

Definition :- The Lagrange polynomials for the data points $\{(x_i, y_i) \mid 0 \leq i \leq n\}$ are given by

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

$(i \in \{0, \dots, n\}), l_i: \mathbb{R} \rightarrow \mathbb{R}$

Note: for $0 \leq i \leq n$, each $l_i(x)$ is a polynomial of degree n

$$l_i(x_k) = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases}$$

$\mathcal{P}_n(\mathbb{R})$ = set of all polynomials, $\mathcal{P}: \mathbb{R} \rightarrow \mathbb{R}$ of degree at most n — vector space. Δ Identity Basis elements

• Interpolating polynomial $P: \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$P(x) = \sum_{i=0}^n q_i l_i(x)$$

[already seen in Example, $n=1$ case]

- let $n \geq 1$
- let $\{x_i, 0 \leq i \leq n\}$ $x_i \in [a, b]$ $x_i \neq x_j$ ($i \neq j$)
- let $p: [a, b] \rightarrow \mathbb{R}$ be the ^{interpolating} Lagrange Polynomial for the data points $\{(x_i, f(x_i)), 0 \leq i \leq n\}$

Take $\int_a^b p(t) dt$ as an approximation

for $\int_a^b f(x) dx$

$$\int_a^b p(t) dt = \sum_{i=0}^n f(x_i) \int_a^b l_i(t) dt$$

$$\approx \sum_{i=0}^n f(x_i) \int_a^b l_i(t) dt$$

Question - How good is the approximation?

Before answering let us look at some examples

• Rectangle rule

$$n=0 \quad X = \{a\}$$

Lagrange Polynomial

$$p(t) = f(a) \quad t \in [a, b]$$

$$\int_a^b p(t) dt = (b-a)f(a)$$



Error bounds

f is $n+1$ times differentiable

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$X = \{ (x_i, f(x_i)) \mid i=0, \dots, n \}$$

$p(\cdot)$ [Lagrange] interpolating polynomial
or any other $x_i \neq x_j \quad i \neq j$

$$p(x_i) = f(x_i) \quad i=0, \dots, n \quad \begin{matrix} \text{degree} \\ \text{at most } n \end{matrix}$$

Fix $x \neq x_i$;

$$\text{let } q(t) = f(t) - p(t) = [f(x) - p(x)] \prod_{i=0}^n \frac{(t - x_i)}{(x - x_i)}$$

At $t = x_k$

$$q(x_k) = f(x_k) - p(x_k) = 0 = 0, \quad 0 \leq k \leq n.$$

$$t = x \text{ \& } q'(x) = 0$$

$g^x(\cdot)$ has $n+2$ distinct zeros

Inductively use Rolle's Theorem to conclude

$$\exists \xi = \xi(x, x_0, \dots, x_n) \left\{ \begin{array}{l} \text{lies between} \\ x \in x_0, \dots, x_n \end{array} \right.$$

$$g^{(n)}(\xi) = 0$$

$$\Rightarrow 0 = f^{(n)}(\xi) - 0 = [f(x) - p(x)] \frac{d^{n+1}}{dx^{n+1}} \left(\prod_{i=0}^n \frac{x - x_i}{x - x_i} \right)$$

(*)

Observe

$$\prod_{i=0}^n \left(\frac{t - x_i}{x - x_i} \right) = t^{n+1} \prod_{i=0}^n \frac{1}{x - x_i} + h(t)$$

Polynomial of degree at most n

$$\frac{d^{n+1}}{dt^{n+1}} \left(\prod_{i=0}^n \frac{t - x_i}{x - x_i} \right) = \frac{(n+1)!}{\prod_{i=0}^n (x - x_i)} \quad - (*)$$

From $(*)$ - after rearranging & substituting $(**)$ $\left[\begin{array}{l} f(x_i) = p(x_i) \text{ \& } \\ \text{Convention: } \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i) \bigg|_{x=x_i} = 0 \end{array} \right]$

$$f(x) = p(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i) ; \xi \text{ - between } x_0, \dots, x_n$$