

$$\textcircled{1} \quad x_{n+1} - x_n = \frac{-f(x_n)}{f[x_{n-1}, x_n]}$$

Place in  $\textcircled{xx}$

$$(x_{n+1} - x_n) f[x_{n-1}, x_n] - (c - x_n) f[x_{n-1}, x_n] = \frac{f''(\xi)}{2} (c - x_{n-1})(c - x_n)$$

$$\Rightarrow \left| (x_{n+1} - c) = \frac{f''(\xi)}{2 f[x_{n-1}, x_n]} (c - x_{n-1})(c - x_n) \right|$$

Secant  
method

$$\textcircled{1} \quad x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]}$$

[Well-defined]

$$p(x) = f(x_n) + (x - x_n) f[x_{n-1}, x_n]$$



$$\begin{aligned}
 f(x) - p(x) &= f(x) - f(x_n) - (x - x_n) f[x_{n-1}, x_n] \\
 &= \left[ \frac{f(x) - f(x_n)}{x - x_n} - \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n} \right] (x - x_n) \\
 &= f[x, x_n, x_{n-1}] (x - x_{n-1}) (x - x_n)
 \end{aligned}$$

Worksheet  
on last  
Tuesday

$$\frac{1}{2} f''(\xi) (x - x_{n-1}) (x - x_n)$$

$\xi$  is an interval  
containing  $x, x_n, x_{n-1}$

(\*)



Note : - We have shown  $x, a, b \in f: \mathbb{R} \rightarrow \mathbb{R}$  is twice differentiable  

$$f(x) - f(a) - (x-a)f'(b, a) = \frac{f''(\xi)}{2} (x-b)(x-a)$$

$$\xi \in \text{Interval that contains } \{x, a, b\}$$

Apply  $x = c$  in (\*)

$$0 - p(c) = \frac{f''(\xi)}{2} (c-x_{n-1})(c-x_n)$$

(\*\*) 
$$0 - f(x_n) - (c-x_n)f'(x_{n-1}, x_n) = \frac{f''(\xi)}{2} (c-x_{n-1})(c-x_n)$$



Fixed point approach

$\exists \alpha < 1$   
for  $x, y \in \mathbb{R}$

$g: \mathbb{R} \rightarrow \mathbb{R}$

$$|g(x) - g(y)| < \alpha |x - y|$$

Aim, - Find  $c \in \mathbb{R}$  such that

$$c = g(c)$$

Relevance -

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$c = g(c) \iff f(c) = 0$$



$$x_0 \in \mathbb{R}$$

$$x_n = g(x_{n-1}) \quad \forall n \geq 1$$

• If at some  $k \in \mathbb{N}$   $x_k = x_{k-1}$  then we are done

• If  $x_k \rightarrow c$  as  $k \rightarrow \infty$  & ( $g$  is continuous)

$$x_k = g(x_{k-1}) \Rightarrow c = g(c)$$

$\downarrow$   $\downarrow$   
 $c$   $c$   
 as  $k \rightarrow \infty$

&  $g$  is continuous

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})|$$

$$\leq \alpha |x_n - x_{n-1}|$$

• Inductively we can get

$$|x_{n+1} - x_n| \leq \alpha^{n-1} |x_2 - x_1| \quad \forall n \geq 1$$



Check if sequence is a Cauchy Sequence

$n \geq m$

$x_n - x_m$  — need to understand its behavior

$$|x_n - x_m| \leq \sum_{k=1}^{n-m} |x_{m+k} - x_{m+k-1}|$$

Triangle  
inequality

Induction

$$\leq |x_2 - x_1| \sum_{k=1}^{n-m} d^{m+k-2}$$

$$= \frac{|x_2 - x_1|}{d^2} d^m \left( \sum_{k=1}^{n-m} d^k \right)$$



$$1. \quad g: \mathbb{R} \rightarrow \mathbb{R} \quad \exists \ 0 < \alpha < 1 \quad \text{such that } |g(x) - g(y)| < \alpha |x - y|$$

$$\in \lambda, y \in \mathbb{R} \quad \text{a. continuous take}$$

$$\left\{ \begin{array}{l} \text{let } \delta > 0. \quad \delta = \frac{\alpha}{2} \\ |x - y| < \delta \\ \Rightarrow |g(x) - g(y)| < \alpha \delta \end{array} \right.$$

$$x_1 \in \mathbb{R} \quad x_n = g(x_{n-1}) \quad \forall n \geq 2$$

Observe from  $(+ + x)$

$$\underline{n > m}$$

$$|x_n - x_m| < \frac{|x_2 - x_1|}{\alpha^2} \alpha^m \sum_{k=1}^{n-m} \alpha^k$$

As  $0 < \alpha < 1$  then  $\frac{\alpha}{1-\alpha} = \sum_{k=1}^{\infty} \alpha^k \neq \sum_{k=1}^{n-m} \alpha^k < \frac{\alpha}{1-\alpha} \quad \begin{matrix} m \geq 1 \\ n \geq 1 \end{matrix}$

$$\Rightarrow |x_n - x_m| \leq \frac{|x_2 - x_1|}{\alpha(1-\alpha)} \alpha^m \quad n \geq 1, m \geq 1 \text{ and } n > m$$



$$\alpha = \frac{1}{1+\delta} \text{ for some } \delta > 0 \quad \& \quad \alpha^m < \frac{1}{m\delta} \quad \forall m \geq 2$$

$$\underline{N \geq m \geq 1}$$

$$|x_n - x_m| < \frac{|x_2 - x_1|}{\alpha(1-\alpha)\delta} \cdot \frac{1}{m}$$

[induction]

Let  $\varepsilon > 0$  be given.

$$N > \frac{\alpha(1-\alpha)\delta}{|x_2 - x_1|} \cdot \frac{1}{\varepsilon} + 1$$

$$m \leq N \Rightarrow |x_n - x_m| < \frac{|x_2 - x_1|}{\alpha(1-\alpha)\delta} \cdot \frac{1}{N} < \varepsilon$$

As  $\varepsilon > 0$  was arbitrary  
the sequence is a  
Cauchy sequence.

$$\Rightarrow \exists c \in \mathbb{R} \quad x_n \rightarrow c \text{ as } n \rightarrow \infty$$

&  $g$  is continuous

$$\Rightarrow g(c) = c$$