Fixed point approach

$g : \mathbb{R} \to \mathbb{R}$, $x_1 \in \mathbb{R}$ & $x_{n+1} = g(x_n)$ $\forall n \geq 1$

Assume : $x_n \neq g(x_{n-1})$ $\forall n \geq 1$

Relevance :- $g(x) = x - \dfrac{f(x)}{f'(x)}$ [Provided well defined]

could answer Newton's Method calculation.

Assume :- $g \in \mathrm{Lip}(\alpha)$ i.e.

$|g(x) - g(y)| < \alpha |x-y|$ $\forall x, y \in \mathbb{R}$

$\forall n \geq 2$;

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})|$$

$$\leq \alpha |x_n - x_{n-1}|$$

$$\text{(inductively)} \leq \alpha^{n-1} |x_2 - x_1|$$

$n, m \in \mathbb{N}$ & $n \geq m$

$$\therefore |x_n - x_m| \leq \sum_{k=1}^{n-m} |x_{m+k} - x_{m+k-1}|$$

$$\leq |x_2 - x_1| \sum_{k=1}^{n-m} \alpha^{m+k-2}$$

$$= |x_2 - x_1| \alpha^{m-2} \sum_{k=1}^{n-m} \alpha^k$$

If $0 < \alpha < 1$;

$$\sum_{k=1}^{n-m} \alpha^k \leq \frac{\alpha}{1-\alpha}$$

$n \geq m$

$$|x_n - x_m| \leq \frac{|x_2 - x_1|}{\alpha(1-\alpha)} \alpha^m .$$

[Assume $\alpha = (1+\delta)^{-1}$ $\delta > 0$; $(1+\delta)^m \geq m\delta$]

$$\Rightarrow \quad |x_n - x_m| \leq \frac{|x_2 - x_1|}{\alpha(1-\alpha)} \cdot \frac{1}{m\delta}$$

let $\varepsilon > 0$ be given $N > \frac{\alpha(1-\alpha)\delta}{|x_2 - x_1|} \cdot \frac{1}{\varepsilon}$

$n \geq m \geq N \Rightarrow$

$$|x_n - x_m| < \varepsilon .$$

As $\varepsilon > 0$ was arbitrary this implies

$\langle x_n \rangle_{n \geq 1}$ is a Cauchy Sequence

$\Rightarrow \quad x_n \longrightarrow c$ as $n \to \infty$.

g is continuous

[$\because$ Given $\varepsilon > 0$ let $\delta = \frac{\varepsilon}{8}$ $\Rightarrow$ $|x-y| < \delta$ then $|g(x) - g(y)| < \varepsilon$]

$\therefore \quad g(x_n) \to g(c) \quad \text{as } n \to \infty.$

$\& \quad x_{n+1} \to c \quad \text{as } n \to \infty$

$\Rightarrow \quad c = g(c) \quad ! \quad \square$

[fixed point]

<u>Issues</u>:- . N ≡ rate of convergence depends on α & x_2, x_1

. g is twice differentiable

$$g(x) = g(c) + (x-c)\,g'(c) + \frac{(x-c)^2}{2} g''(\xi)$$

ξ - between x and c.

Suppose $g'(c) = 0$

Then $g(x_n) = c + \frac{(x_n-c)^2}{2} g''(\xi)$

$$\Rightarrow \quad x_{n+1} - c = \frac{(x_n-c)^2}{2} g''(\xi)$$

Suppose $g''(\cdot)$ is bounded

$$\Rightarrow \quad |x_{n+1} - c| \leq M\, |x_n - c|^2$$

we have quadratic convergence

Newton's Method :-

$$g(x) = x - \frac{f(x)}{f'(x)}$$

if well defined

- $g(c) = c, \quad f'(c) \neq 0 \Rightarrow f(c) = 0$

- $g'(x) = 1 - \cdots$

$$= \frac{f(x)\, f''(x)}{(f'(x))^2}$$

$f(c) = 0 \Rightarrow g'(c) = 0$

- $g''(x) = \cdots\cdots =$

$$= \frac{f''(x)}{f'(x)} + \frac{f(x)}{f'(x)^3}\left[f'(x) f'''(x) - 2 f''(x)^2\right]$$

$$g''(c) = \frac{f''(c)}{f'(c)}$$

$$\Rightarrow \quad x_{n+1} - c = \frac{g''(\xi)}{2}(x_n - c)^2$$

n - large $\quad x_{n+1} - c$ "is close to" $\frac{1}{2}\left|\frac{f''(c)}{f'(c)}\right|(x_n - c)^2$

$f : \mathbb{R}^n \to \mathbb{R}^n \qquad x \in \mathbb{R}^n \longrightarrow (f_i(x))_{1 \le i \le n}$

$f_i : \mathbb{R}^n \to \mathbb{R}$

Newton's Method :-

$$x_{n+1} = x_n - [Df(x_n)]^{-1} f(x_n)$$

$$Df(y) = \left[\frac{\partial f_i}{\partial x_j}(y) \right]_{\substack{1 \le i \le n \\ 1 \le j \le n}}$$

- Verify geometrical interpretation
- Work with Taylor series in higher dimensions
- lay down assumptions.

Show : $x_n \to c$ with $f(c) = 0$

Optimization :-

$F : \mathbb{R}^n \to \mathbb{R} \qquad \max_x F(x)$ or $\min_x F(x)$

Need to find c : $DF(x) = 0$.

Assume : $D^2(F(x)) > 0$ positive definite

Apply Newton's Method:-

- choose x_0:

$\forall n \geq 1$ $$x_n = x_{n-1} - [\nabla\nabla F(x_n)]^{-1} \nabla F(x_n) \quad -(1)$$

$\nabla^2 F(\cdot)$ Hessian of F

One can develop suitable criteria for convergence of $\{x_n\}_{n \geq 1}$. - for close enough starting points

Gradient descent:-

$$x_n = x_{n-1} - a \nabla F(x_n)$$

for some sufficiently small 'a'.

- One could allow a to depend on 'n'.
- Converges slower than Newton's method
- for a larger class of initial guesses.