Taylor's Theorem : Let $a, x \in \mathbb{R}$ & $f : \mathbb{R} \to \mathbb{R}$ $(n+1)$ times differentiable function (only require on an open interval containing a and x) Then

$$f(x) = P_n(x) + R_n(x)$$

where
$$P_n(x) = \sum_{k=1}^{n} f^{(k)}(a) \frac{(x-a)^k}{k!} + f(a)$$

$$R_n(x) = f^{n+1}(c) \frac{(x-a)^n}{(n+1)!}$$

with $f^{(k)}$ – being the k^{th} derivative of f & c a point between x and a.

Proof :- Assume $a < x$. are fixed.
let $n \geq 1$ be given

$$F : [a, x] \to \mathbb{R}$$

$$F(t) = f(x) - f(t) - \sum_{k=1}^{n} f^{(k)}(t) \frac{(x-t)^k}{k!}$$

Note :- $F(x) = 0$

$$F(a) = f(x) - f(a) - \sum_{k=1}^{n} f^{(k)}(a) \frac{(x-a)^k}{k!} \quad -(1)$$

$t \in (x,a)$: $F'(t) = -\dfrac{f^{(n+1)}(t)}{n!}(x-t)^n$

[by induction]

$G : [x,a] \to \mathbb{R}$

$$G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a)$$

$G(x) = 0$ — G – continous in $[x,a]$

$G(a) = 0$ — differentiable in (x,a)

Rolle's Theorem $\exists$ $c \in (x,a)$

$$G'(c) = 0$$

$\Rightarrow$

$$0 = F'(c) + (n+1)\frac{(x-c)^n}{(x-a)^{n+1}} F(a)$$

$$= -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + \frac{(n+1)(x-c)^n}{(x-a)^{n+1}} F(a)$$

$\Rightarrow$

$$F(a) = \frac{(x-a)^{n+1} f^{n+1}(c)}{(n+1)!} \quad - (2)$$

From 1 and 2

$$f(x) - f(a) - \sum_{k=1}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(c)$$

$\Rightarrow$

$$f(x) = T_n(x) + R_n(x).$$

Similarly one can write a proof for the case $a < x$ by considering

$$\tilde{F}(t) = f(x) - f(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)(t-x)^k}{k!}$$

$$\tilde{g}(t) = \tilde{F}(t) - \left(\frac{t-x}{a-x}\right)^{n+1} \tilde{F}(a)$$

So we are done. $\square$

Remark: Let $0 < h$ and $a \in \mathbb{R}$.
$f : [a-h, a+h] \to \mathbb{R}$ be only differentiable, with all derivatives being bounded by M.

For $\forall n \geq 1$ $x \in (a-h, a+h)$, $|R_n(x)| \leq \dfrac{M \cdot h^{n+1}}{(n+1)!}$.

let $\epsilon > 0$ be given. let $x \in (a-h, a+h)$

$K \in \mathbb{N}$ s.t. $h < K$

$\exists C > 0$ s.t. $\forall n \geq K+1$ $\quad$ $\boxed{\text{by induction}}$

$(n+1)! \geq C K^n$

$\left[(n+1)! \geq K! \, K^{n-K} = \left(\dfrac{K!}{K^K}\right) K^n \geq C K^n \right]$

$\Rightarrow \quad |R_n(x)| < \dfrac{M \cdot h}{C} \left(\dfrac{h}{K}\right)^n = C_1 \left(\dfrac{h}{K}\right)^n$

$\dfrac{h}{-} < 1 \Rightarrow \dfrac{h}{2} = \dfrac{1}{1+\beta}$, for some $\beta > 0$

& $(1+\beta)^n \geq 1 + n\beta$ [Binomial expansion]

$\Rightarrow \quad |R_n(x)| < \dfrac{C_1}{1+n\beta}$

Choose $N \geq 1$: $\dfrac{C_1}{1+n\beta} < \varepsilon$

So for: $n \geq N$ $\quad |R_n(x)| < \varepsilon$

As $\varepsilon > 0$ and $x \in (a-h, a+h)$ was arbitrary.

$R_n(x) \longrightarrow 0$, uniformly in $x \in (a-h, a+h)$, as $n \to \infty$ $\square$

<u>Example</u> :-

$f(x) = e^x$

<u>Real - Analysis (recall)</u>

Define :- • $e := \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ or $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!}$

• e^k $k \in \mathbb{N}$

• e^z $z \in \mathbb{Z}$

• $e^{p/q}$ $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $q \neq 0$

• $e^x = \sup\{e^r \mid r \leq x ; r \in \mathbb{Q}\}$ $x \in \mathbb{Q}^c$

- differentiable ; ·—····· $(e^x)' = e^x$

$a = 0$;
Taylor's Thm.
$\forall n \geq 1$
c - between $0, x$

$$e^x = \underbrace{1 + \sum_{k=1}^{n} \frac{x^k}{k!}}_{P_n(x)} + \underbrace{e^c \frac{x^{n+1}}{n+1!}}_{R_n(x)}$$

One views, $P_n(x)$ as a truncated approximation of e^x.

Also, $R_n(x)$ as the Truncation error.

Observe $-M < x < M \Rightarrow$

$$|R_n(x)| < \frac{e^M M^{n+1}}{(n+1)!}$$

We will see in R-code how Truncation error and Round-off error in computing $P_n(x)$ interact.

<u>Order Computation</u> :- "Big O"

Let $a: \mathbb{N} \to \mathbb{R}$ and $b: \mathbb{N} \to \mathbb{R}_+$ be two sequences. We say

(i) $a_n = O(b_n)$ (as $n \to \infty$ - usually omitted)

if $\exists\ c_1 > 0$ and $N \geq 1$ such that

$$|a_n| \leq c_1 b_n \quad \forall n \geq N$$

$f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ $\quad g : \mathbb{R} \to \mathbb{R}_+$ be two functions

(ii) $f(x) = O(g(x))$ as $x \to \infty$ if

$\exists\, C_1 > 0$, $M > 0$: $|f(x)| < C_1 g(x) \quad \forall x > M$.

(iii) $f(x) = O(g(x))$ as $x \to a$ if

$\exists\, C_1 > 0$ & $\delta > 0$

: $|f(x)| < C_1 g(x) \quad \forall x \in (a-\delta, a+\delta)$

Typically (ii) is used at $a = 0$.

Examples :-

$\alpha_n :=$ represents the # of computations in Horner's method for a n^{th} degree polynomial.

Horner's Method :- Rewrite

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

$$= a_0 + a_1 x + \cdots + x^{n-1}(a_{n-1} + x a_n)$$

$$\vdots$$

$$= a_0 + x(a_1 + \cdots + x(a_{n-1} + x a_n))$$

we know if $\beta_n = n$ then

$$\alpha_n = O(\beta_n).$$

This is simply written as $\alpha_n = O(n)$

$\square$

<u>Example</u> : $a_n = n^2 + 2n + 5$

$$a_n = n^2\left(1 + \frac{2}{n} + \frac{5}{n^2}\right)$$

$n \geq 4$ $\quad 1 + \frac{2}{n} + \frac{5}{n^2} < 3$

let $b_n = n^2$

$n \geq 4 \Rightarrow a_n < 3\, b_n.$

$\therefore$ $a_n = O(b_n)$ or simply written as

$$a_n = O(n^2).$$

Example:- $f(x) = \sqrt{x}$

$a = 4 \qquad f(a) = 2$

$f'(a) = \frac{1}{2 \cdot 2} = \frac{1}{4}$

x - near a

$$f(x) = 2 + (x-a)\cdot\frac{1}{4} + \frac{f''(\xi)(x-a)^2}{2}$$

let $x = a+h \qquad |h| < 1$

$$f(a+h) = 2 + \frac{h}{4} + \frac{f''(\xi)\,h^2}{2}$$

$$R_1(h) = O(h^2) \quad \text{as } h \to 0$$

with

$0 < f''(\xi) < \frac{1}{2}$. You can use this approximation to calculate square root of say 3.82 with a clear error bound.

Example: Finite difference approximations

Assuming $f: \mathbb{R} \to \mathbb{R}$ satisfies Taylor's Theorem hypothesis; we have

$$f(a+h) = f(a) + h f'(a) + \frac{f''(\xi)\,h^2}{2}$$

ξ is between a and $a+h$.

$$\Rightarrow \quad f'(a) = \frac{f(a+h) - f(a)}{h} + \frac{R_1(h)}{h}$$

with $R_1(h) = O(h^2)$ if second derivative was bounded.

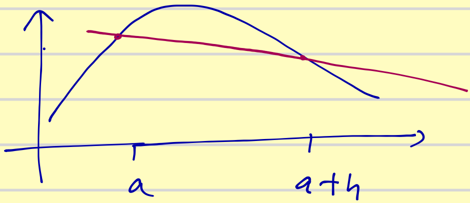
$$\Rightarrow \quad f'(a) = \frac{f(a+h) - f(a)}{h} + O(h)$$

- Depending on $h > 0$ or $h < 0$, we are approximating $f'(a)$ from one side of a

Notation means:

$$\frac{f(a+h) - f(a)}{h} + g(h)$$

where $g(h) = O(h)$ as $h \to 0$



Now $h > 0$,

$$f(a+h) = f(a) + f'(a)h + f''(\xi_1)\frac{h^2}{2}$$

$$f(a-h) = f(a) - f'(a)h + f''(\xi_2)\frac{h^2}{2}$$

with $\xi_1 \in (a, a+h)$

$\xi_2 \in (a-h, a)$

$$\Rightarrow \quad f(a+h) - f(a-h) = 2f'(a)h + \tilde{R}(h)$$

where

$$\tilde{R}(h) = \left[f''(\xi_1) - f''(\xi_2)\right]\frac{h^2}{2} \underset{\text{M.V.T.}}{=} f'''(\xi_3)(\xi_1 - \xi_2)\frac{h^2}{2}$$

$$\Rightarrow \quad \tilde{R}(h) = O(h^3) \quad \text{if } f'''(\cdot) \text{ is bounded}$$

$$\therefore \quad f'(a) = \frac{1}{2h}\left(f(a+h) - f(a-h)\right) + \frac{\tilde{R}(h)}{h}$$

$$\Rightarrow \quad f'(a) = \frac{f(a+h) - f(a-h)}{2h} + \underbrace{O(h^2)}_{\text{redaction in error}}$$

Say $f(a+h) = f^{FD}(a+h) + e_1$

$f(a-h) = f^{FD}(a-h) + e_2$

$$f'(a) = \frac{f^{FD}(a+h) - f^{FD}(a-h)}{2h} + \frac{e_1 - e_2}{2h} + O(h^2)$$

Round off error increases as $h \to 0$

Truncation error decreases as $h \to 0$