(*Divided Difference*) Let $f : [a, b] \to \mathbb{R}$ be two times differentiable on $[a, b]$. Define for $x, y, z \in (a, b)$ with $x \neq y \neq z$,

$$f[x] = f(x_0)$$
$$f[x, y] = \frac{f[x] - f[y]}{x - y} = \frac{f(x) - f(y)}{x - y}$$
$$f[x, y, z] = \frac{f[x, y] - f[y, z]}{x - z} = \frac{1}{x - z}\left(\frac{f(x) - f(y)}{x - y} - \frac{f(y) - f(z)}{y - z}\right)$$

Show that there exists $\xi \in (a, b)$ such that

$$f[x, y, z] = \frac{f''(\xi)}{2}.$$

[1]

1. (*Polynomials*) In R, there is an in-built function that finds roots of polynomials(both real and complex coefficient), called polyroot. It uses the Jenkins and Traub (1972), TOMS Algorithm 419. Comm. ACM, 15, 9799. Suppose we were to find the roots of $3x^2 + 5x + 2$ then in R

```
> polyroot(c(2,5,3))

[1] -0.6666667+0i -1.0000000-0i
```

There is another package called PolynomF.

```
> require("PolynomF")
> x = polynom() # it saves x as an object Polynomial
> W = (x-1)(x-2)
> W

-3 + x

> solve(W)

[1] 3

> #or
> (q = solve(2*x^2 -6*x +4))

[1] 1 2

> poly_calc(q) # calculates it back from root normalised

2 - 3*x + x^2
```

Consider the Polynomial given by $1 + 2ix + (3 - 7i)x^2$ Find its root using polyroot function. Verify for yourself that solve using the package PolynomF does not work.

[1] Hint: Use $g : [a, b] \to \mathbb{R}$ given by $g(w) = f(w) - f[x] - f[x, y](w - x) + f[x, y, z](w - x)(w - y)$

2. (*Wilkinson Polynomial*)[2] It is easy to see what the roots of the polynomial given by

$$W(x) = \prod_{i=1}^{20}(x - i).$$

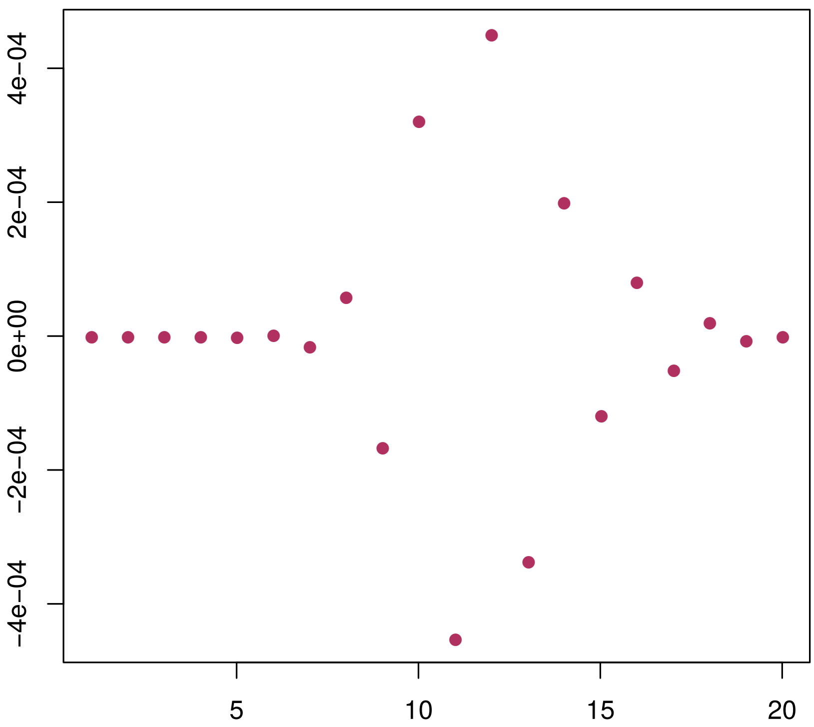
are. Using the R-code given in the file wilk.R construct the polynomial W in R. Using the solve[3] and polyroot find the roots of W. Comment on the solution

3. It is easy to see what the roots of the polynomial given by

$$V(x) = (x - 1)^{100}.$$

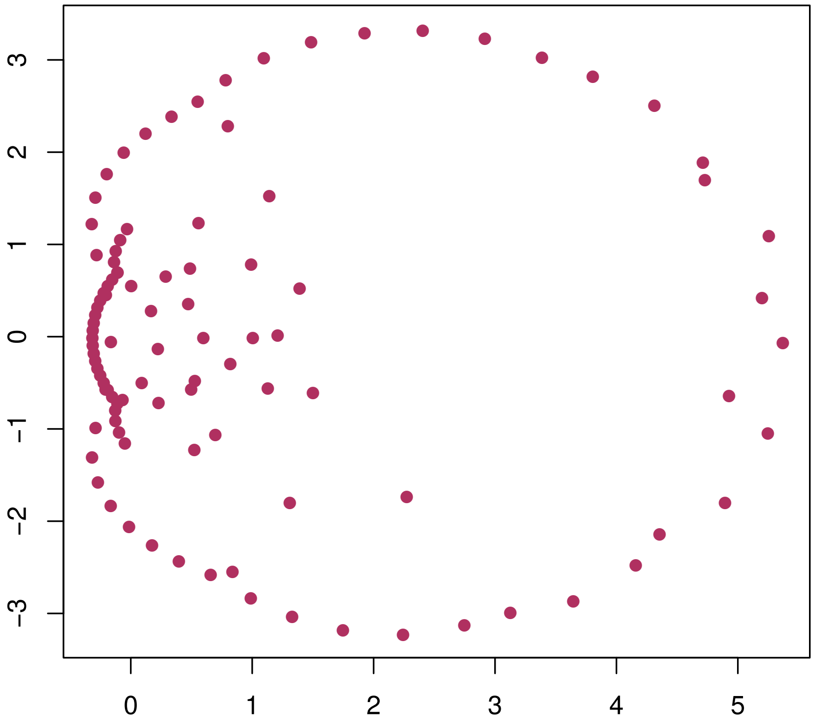
are. Write an R-code to construct the polynomial V in R. Using the polyroot find the roots of V. Comment on the solution

4. Using the work done in this worksheet can produce the below pictures:



[2] Recall your Analysis I assignment on the fact about this polynomial having distinct minima. Here is an elegant solution of that by Partha Pratim Ghosh of ISI Delhi.

[3] uses an eigen value computation as detailed by this note.



Please explain the phenonmena. Not an immediate connection but do view Chladni plates and the blog on Chladni Figures

5. We wish to find the non-zero root of the function $f : [-\pi, \pi] \to \mathbb{R}$ given by

$$f(x) = x^3 - (\sin(x))^2.$$

   (a) Plot the function, along with the horizontal line $y = 0$ to identify the zeros.

   (b) Using the code bisect.R implement the bisection method and find a range for the zero when tolerance for the interval size given is by tol =1e-5.

   (c) Using the code newton.R implement the Newton-Raphson method and find the zero when convergence criteria is decided by the iterate difference below tol =1e-5 or the maximum number of iterations exceeds 50.

   (d) Decide which method performs better.

Due Date: February 6th, 2020.
Problems due: 3,4

1. Finish the inclass worksheet.
2. Write a function in R called secant.R and implement the Secant method and find the zero when convergence criteria is decided by the iterate difference below tol =1e-5 or the maximum number of iterations exceeds 50. Use it to find the non-zero root of the function $f : [-\pi, \pi] \to \mathbb{R}$ given by

$$f(x) = x^3 - (\sin(x))^2.$$

   Compare your answer with the Bisection method and Newton-Raphson method done in the worksheet.
3. For each of the function $f : [-2, 2] \to \mathbb{R}$ given by

$$(i) f(x) = x^3 - x \qquad (ii) f(x) = x^2 - x.$$

   The functions have multiple roots.

   (a) Implement the Bisection method via the code bisect.R as in the worksheet and see if there is a preferred root chosen by the method.

   (b) Implement Newton-Raphson method via the code newton.R as in the worksheet for both functions with various starting points and see if there is convergence always and if so is there a preferred root.
4. Let $f : \mathbb{R} \to \mathbb{R}$ and f be twice differentiable with continuous second derivative. Suppose
   - there is a $c \in \mathbb{R}$ such that $f(c) = 0$.
   - there[4] is a $\delta > 0$, such that $f'(x) \neq 0$ for all $x \in (c - \delta, c + \delta)$

   Show that there is an $\eta > 0$ such that if $x_0 \in (c - \eta, c + \eta)$ then

   (a) for all $k \geq 1$,

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$$

   is a well defined sequence of real numbers;

   (b) there is a $M > 0$ such that

$$| x_k - c | \leq M(x_{k-1} - c)^2$$

   for all $k \geq 1$; and

   (c) $x_k \to c$ as $k \to \infty$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ and f be twice differentiable with continuous second derivative. Suppose
   - there is a $c \in \mathbb{R}$ such that $f(c) = 0$.
   - there[4] is a $\delta > 0$, such that $f'(x) \neq 0$ for all $x \in (c - \delta, c + \delta)$

   Show that there is an $\eta > 0$ such that if $x_0 \in (c - \eta, c + \eta)$ then

   (a) for all $k \geq 1$,

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f[x_{k-1}, x_{k-2}]}$$

   is a well defined sequence of real numbers;

   (b) there is a $M > 0$ such that

$$| x_k - c | \leq M \mid x_{k-1} - c \mid^p$$

   for all $k \geq 1$ with $p = \frac{1+\sqrt{5}}{2}$; and

   (c) $x_k \to c$ as $k \to \infty$.

[4] It is enough to assume that $f'(c) \neq 0$.