1. Let $a > 0$ and choose $s_1 > \sqrt{a}$. Define

$$s_{n+1} := \frac{1}{2}(s_n + \frac{a}{s_n})$$

for $n \in \mathbb{N}$.

(a) Show that $\lim_{n\to} s_n = \sqrt{a}$.

(b) If $z_n = s_n - \sqrt{a}$ then show that $z_{n+1} < \frac{z_n^2}{2\sqrt{a}}$.

Solution: Note that $s_1 \geq \sqrt{a}$ by assumption. Let us assume that for $n = k$ that $s_k \geq \sqrt{a}$. Then, with $a > 0$, it is immediate that $s_k > 0$. Using the fact that Arithmetic mean is greater than geometric mean we have

$$s_{k+1} = \frac{1}{2}(s_k + \frac{a}{s_k}) \geq \sqrt{s_k \times \frac{a}{s_k}} = \sqrt{a}.$$

Thus by induction that $s_n \geq \sqrt{a}$ for all $n \geq 1$. For $n \geq 1$, we now have

$$s_{n+1} - s_n = \frac{1}{2}(s_n + \frac{a}{s_n}) - s_n = \frac{a - s_n^2}{2s_n} < 0 \qquad (\text{as } 0 < \sqrt{a} < s_n).$$

This implies that s_n is a decreasing sequence. Since it is bounded below by $\sqrt{a}$, the sequence converges to a number s with $s \geq \sqrt{a}$. As s_{n+1} is a subsequence of s_n it also converges to s as $n \to \infty$. With $s \neq 0$ and each $s_n > 0$, we have $\frac{1}{s_n} \to \frac{1}{s}$. Letting $n \to \infty$ on both sides of

$$s_{n+1} = \frac{1}{2}(s_n + \frac{a}{s_n})$$

and using the algebra of limits we have

$$s = \frac{1}{2}(s + \frac{a}{s}).$$

As $s \geq \sqrt{a}$, this will imply that $s = \sqrt{a}$. Therefore we have shown that $s_n \to \sqrt{a}$ as $n \to \infty$. □

2. (Due at 345pm in a Rnw, pdf file in dropbox subfolder week1)

(a) What is the following R-code attempting to do:

```
> function(a,n,tol){
+ deltax = 1
+     while (abs(deltax) > tol){
+         deltax = (1/n)*(a /x^(n-1) -x)
+         x = x + deltax
+     }
+     return(x)
+ }
```

Explain the lines using comments in the Rnw file.

Solution:Fix $n \geq 1$, let $s_1 = a$ with $a > 0$. For each $k \geq 1$ consider t

$$s_{k+1} = \frac{1}{n}\left((n-1)s_k + \frac{a}{s_k^{n-1}}\right).$$

It can be shown that as $k \to \infty$, s_k goes to n-th root of a. The R-code below executes the above sequence and stops when $| s_{k+1} - s_k |$ is smaller than a specified tolerance value tol.

```
> nthrootofa=function (a,n,tol){
+        #function calculates n-th root of the number 'a'
+        # iterative scheme is stopped when successive differences
+        # are below a specified tolerance level.
+        #'tol'is the tolerance level
+
+ deltax=1  #set initial iterative difference to be 1
+ x=a       #value of s_1
+ while(abs(deltax) > tol){
+      #specifies that the loop will run till the absolute value
+      # of difference of successive iterates is less than 'tol'
+
+
+ deltax= (1/n)*(a/(x^(n-1)) -x)
+  #deltax stores the difference of successive
+  #iterates emulating successive difference
+  # of the sequence terms
+
+ x=x+deltax
+  #x stores the next iterate, or in other
+  # words the next term of the sequence.
+ }
+
+ return(x)
+  #returns the value of the last iterate, which is
+  #the n-th root of a at the desired tolerance
+ }
```

□

(b) Can you write a R-code to compute the square root of a upto a tolerance of $\frac{1}{1000}$. Find the square root of $2, \pi, 10$ with this program and also using the sqrt command in R. Save your work in a file called

Solution:We can simply call the function in the (a) at $a = 2, \pi, 10$, $n = 2$ and tol $= \frac{1}{1000}$

```
> nthrootofa(2,2,1/1000)

[1] 1.414214

> nthrootofa(pi,2,1/1000)

[1] 1.772454

> nthrootofa(10,2,1/1000)

[1] 3.162278
```

□

3. Can you formulate the recurssive sequence in question 1 appropriately to say $s_n^{(k)}$ and prove that it converges to the k-the root of a ?

Solution: Fix $n \geq 1$, let $s_1 > \sqrt[n]{a}$ with $a > 0$. For each $k \geq 1$ consider

$$s_{k+1} = \frac{1}{n}\left((n-1)s_k + \frac{a}{s_k^{n-1}}\right). \tag{1}$$

Note that $s_1 \geq \sqrt[n]{a}$ by assumption. Let us assume that for $n = k$ that $s_k \geq \sqrt[n]{a}$. Then, with $a > 0$, it is immediate that $s_k > 0$. Using the fact that Arithmetic mean is greater than geometric mean we have

$$s_{k+1} = \frac{1}{n}((n-1)s_k + \frac{a}{s_k^{n-1}}) \geq \sqrt[n]{s_k^{n-1} \times \frac{a}{s_k^{n-1}}} = \sqrt[n]{a}.$$

Thus by induction that $s_k \geq \sqrt[n]{a}$ for all $n \geq 1$ For $k \geq 1$, we now have

$$s_{k+1} - s_k = \frac{1}{n}\left((n-1)s_k + \frac{a}{s_k^{n-1}}\right) - s_k = \frac{a - s_k^n}{n s_k^{n-1}} < 0 \qquad (\text{as } 0 < \sqrt[n]{a} < s_k).$$

This implies that s_k is a decreasing sequence. Since it is bounded below by $\sqrt[n]{a}$, the sequence converges to a number s with $s \geq \sqrt[n]{a}$. As s_{k+1} is a subsequence of s_k it also converges to s as $k \to \infty$. With $s \neq 0$ and each $s_k > 0$, we have $\frac{1}{s_k} \to \frac{1}{s}$. Letting $k \to \infty$ on both sides of (1) and using the algebra of limits we have

$$s = \frac{1}{2n}(s + \frac{a}{s^{n-1}}).$$

As $s \geq \sqrt[n]{a}$, this will imply that $s = \sqrt[n]{a}$. Therefore we have shown that $s_n \to \sqrt{a}$ as $n \to \infty$.

□

Exercise: In the above proof we started with $s_1 > \sqrt[n]{a}$. In Problem 2, R-code, we started with $s_1 = a$. Can you modify the proof above to show that indeed the algorithm presented in the code will also converge ?