Computer Science II **January 30th, 2020** **Quiz 3** **Score**

Name: Solution

For each of the following indicate whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$,$f(n) = \Theta(g(n))$,$f(n) = o(g(n))$

1. $f(n) = nb^n$, for $b \in (0, 1)$ and $g(n) = \frac{1}{n^3}$
2. $f(n) = n^{\log_2 \log_2 n}$ and $g(n) = 2(\log_2 n)^{\log_2 n}$

Solution:

We say that:

- $f(n) = O(g(n))$ if there exists $N_0 \in \mathbb{N}$ and $c > 0$ such that $f(n) \leq cg(n)$ for all $n \geq N_0$
- $f(n) = \Omega(g(n))$ if there exists $N_0 \in \mathbb{N}$ and $c > 0$ such that $f(n) \geq cg(n)$ for all $n \geq N_0$
- $f(n) = \Theta(g(n))$ if there exists $N_0 \in \mathbb{N}$ and $c_1, c_2 > 0$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq N_0$
- $f(n) = o(g(n))$ if for every $c > 0$ there exists N_0 such that $f(n) \leq cg(n)$ for all $n \geq N_0$

It is clear that if $f(n) = o(g(n))$. This immediately implies that $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$. Consequently, $f(n) \neq \Theta(g(n))$,

Secondly, if $f(n) = \Theta(g(n))$ this immediately implies that $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$ and $f(n) \neq o(g(n))$.

1. As $b < 1$ there is a $\delta > 0$ such that $b = \frac{1}{1+\delta}$. Now for any $n \geq 1$

$$0 \leq \frac{f(n)}{g(n)} = \frac{nb^n}{\frac{1}{n^3}} = n^4 b^n = \frac{n^4}{(1+\delta)^n}.$$

Observe that $n \geq 6$,

$$(1+\delta)^n = \sum_{k=0}^{n} \binom{n}{k} \delta^k \geq \binom{n}{5} \delta^5 = \frac{\delta^5}{5!} n^5 (1 - \frac{1}{n})(1 - \frac{2}{n})(1 - \frac{3}{n})(1 - \frac{4}{n}) \geq \frac{\delta^5 n^5}{6^4}$$

From both the displays, for $n \geq 5$,

$$0 \leq \frac{f(n)}{g(n)} \leq \frac{6^4}{\delta^5 n}.$$

Let $\epsilon > 0$ be given. Then there exists an $N > 6$ such that $\frac{1}{N} < \frac{\delta^5 \epsilon}{6^4}$. Therefore for all $n \geq N$ we have that

$$0 \leq f(n) < g(n)\epsilon.$$

As $\epsilon > 0$ was arbitrary, we can conclude that $f(n) = o(g(n))$

□

2.Solution: For any $n \geq 1$, we have

$$f(n) = n^{\log_2 \log_2 n} = 2^{(\log_2 n)(\log_2 \log_2 n)} = 2^{(\log_2 \log_2 n)(\log_2 n)} = (\log_2 n)^{\log_2 n} = \frac{1}{2} g(n).$$

Hence $f(n) = \Theta(g(n))$.

□