$\varphi_{n\times n}$

$\text{Range}(\varphi) = \text{Column space of } \varphi.$

$= \{ b_{n\times 1} \mid \exists\, x_{n\times 1} : \varphi x = b \}$ –

$\boxed{\varphi_{*j} \text{ – } j^{th} \text{ column of } \varphi}$

$= \{ b_{n\times 1} \mid x_1 \varphi_{*1} + x_2 \varphi_{*2} + \dots + x_n \varphi_{*n} = b \}$

$= \text{Span}\{ \varphi_{*1}, \varphi_{*2}, \dots, \varphi_{*n} \}$

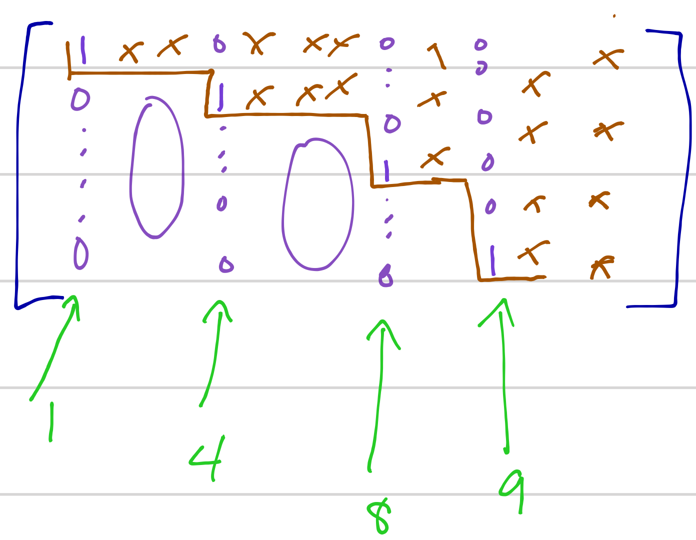
$[\varphi : b] \xrightarrow[\text{operations}]{\text{Row}} [\tilde{U} : \tilde{b}]$

Reduced Echelon form.

- Step 1:- If $x_{n\times 1} : \varphi x = b \Longleftrightarrow \tilde{U}x = \tilde{b}$

Step 2:- "'1' stairs column's will form a basis for column space φ"

$[\tilde{U} : \tilde{b}] =$



$\{ \varphi_{*1}, \varphi_{*4}, \varphi_{*8}, \varphi_{*9} \}$ are basis for column space of φ.

– Basis of column space of φ from the columns of φ.

1 ⓐ

$$Q_{x_1} = v_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

basis for $C(Q)$

$$Q_{x_2} = v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

$$Q_{x_4} = v_4 = \begin{pmatrix} -2 \\ 1 \\ -2 \\ -2 \end{pmatrix}$$

- $\langle v_1, v_2 \rangle = 1 + 0 - 2 + 1 = 0$

  v_1 & v_2 are orthogonal

- $v_4 = \begin{pmatrix} -2 \\ 1 \\ 2 \\ -2 \end{pmatrix}$ $\quad \langle v_2, v_4 \rangle = -2 + 2 + 2 - 2 = 0$

  $\langle v_1, v_4 \rangle = -2 + 0 + 4 - 2 = 0$

$\{v_1, v_2, v_4\}$ orthogonal basis

$v_1, v_2 \qquad v_1 \neq 0, \quad v_2 \neq 0 \qquad v_1 \neq \alpha v_2$ for some α

Set $$p = \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1$$

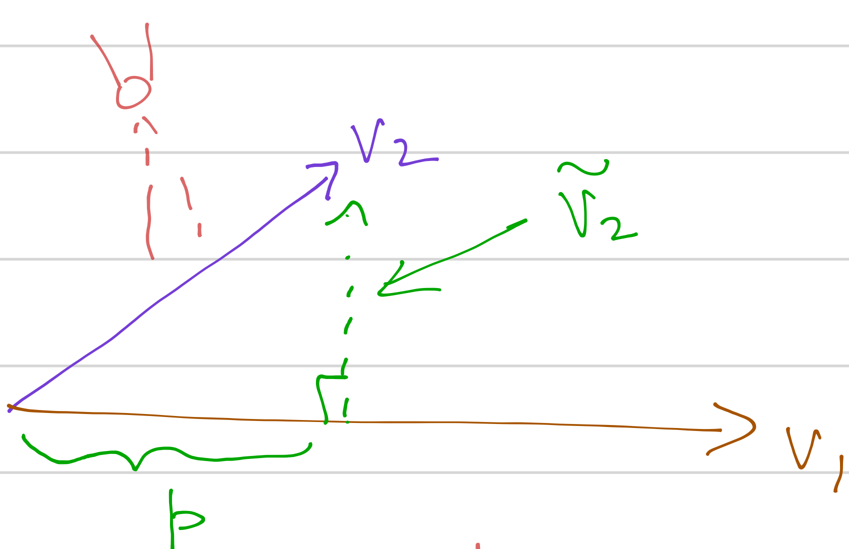
$$\tilde{v}_2 = v_2 - p$$

$$\Rightarrow \langle \tilde{v}_2, v_1 \rangle = \langle v_2, v_1 \rangle - \langle p, v_1 \rangle$$

$$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \langle v_1, v_1 \rangle$$

$$= 0$$

$\{\tilde{v}_2, v_1\}$ are orthogonal



– projection of v_2 onto v_1

· " $\langle a, b \rangle \doteq |a|\,|b| \cos(\theta)$ "

– early 12th class

General:- If $v_1, v_2, \ldots v_k$ are orthogonal

ie $\langle v_i, v_j \rangle = 0$ and $v_i \neq 0$
$i \neq j$

in some vector space V

$x \in V$ & $x \notin \text{Span}\{v_1, v_2 \ldots, v_k\}$

Set $$p = \sum_{i=1}^{k} \frac{\langle x, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

$$v_{k+1} = x - p$$

$\Rightarrow \{v_1, \ldots, v_{k+1}\}$ are orthogonal.

Step 1

φ 1 (c): $[\varphi : \theta]$ - reduced one - $[\tilde{\varphi} : \theta]$

If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix}$ & $\varphi x = 0$

Then $\tilde{\varphi} x = 0$

$$\left.\begin{aligned} &(1) - \quad x_1 + 3x_3 - x_4 - x_6 = 0 \\ &(2) - \quad 7x_2 - 7x_3 - x_4 - 8x_5 + 13x_6 = 0 \\ &(3) - \quad x_4 + x_5 + x_6 = 0 \end{aligned}\right\} - \boxed{\text{Pseudo code ?}}$$

going forward

Step2 :-

$$③ \Rightarrow \quad x_4 = -x_5 - x_6 \qquad - ④$$

$$② \Rightarrow \quad x_3 = x_2 - \frac{[x_4 + 8x_5 - 13x_6]}{7}$$

$$= x_2 - \frac{[7x_5 - 14x_6]}{7}$$

$$= x_2 - x_5 + 2x_6 \qquad - ⑤$$

$$③ \quad x_1 = -3x_3 + x_4 + x_6$$

$$= -3x_2 + 3x_5 - 6x_6$$
$$-x_5 - x_6 + x_6$$

$$= -3x_2 + 2x_5 - 6x_6 \qquad - ⑥$$

$x_{n \times 1} : Ax = 0 \Rightarrow$

$$x = \begin{bmatrix} -3x_2 + 2x_5 - 6x_6 \\ x_2 \\ x_2 - x_5 + 2x_6 \\ -x_5 - x_6 \\ x_5 \\ x_6 \end{bmatrix}$$

$$x_2, x_5, x_6 \in \mathbb{R}$$

$$= x_2 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -6 \\ 0 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\{x : \Phi x = 0\} = \text{Span}\left\{ \overset{v_1 \,\equiv}{\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \overset{v_2 \,\equiv}{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_3 \,\equiv}{\begin{bmatrix} -6 \\ 0 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

verify : Independence ✓

$\Rightarrow \{v_1, v_2, v_3\}$ are a basis for Null(Φ).

<u>Exercise</u> :- Given $\Phi_{m \times n}$:

write an

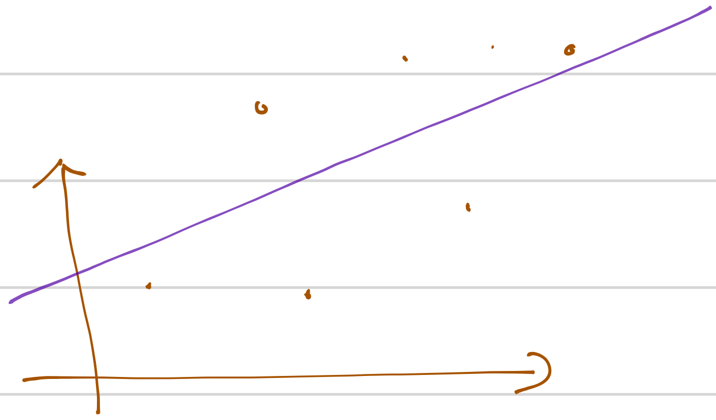
(a) R-code that will display a basis for Column space of Φ from the columns of Φ ... $\boxed{\text{orthogonal basis}}$

(b) R-code - for basis of Null(Φ).

least-squares Method

$x_1 \dots x_n$

$y_1 \dots y_n$



$$y = \alpha + \beta x$$

Solve for α, β:

$$\begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

- System is inconsistent

$$A = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \qquad c = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\arg\min_{c} \|y - Ac\|_2^2 = \arg\min_{\alpha,\beta} \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(c) = \|y - Ac\|_2^2$$
$$= (y - Ac)^T (y - Ac)$$
$$= y^T y - y^T A c - c^T A^T y + c^T A^T A c$$

- Set $\nabla f(c) = 0$ critical points
- orthogonal projections idea

— Exercise $\Rightarrow$ $-2A^T y + 2A^T A c = 0$

$\Rightarrow$ c solves :- $\boxed{A^T A c = A^T y}$ $\equiv$ Normal Equations