Due Date: March 31st, 2020

1. (a) Find a basis for $Range(Q)$, where Q and its reduced form are:

$$Q = \begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & 2 & 4 & 1 \\ -2 & 1 & 0 & -2 \\ 1 & 1 & 3 & -2 \end{pmatrix}; \qquad (Q|b) \to \begin{pmatrix} 1 & 1 & 3 & -2 & b_1 \\ 0 & 2 & 4 & 1 & b_2 \\ 0 & 0 & 0 & -15 & 2b_3 + 4b_1 - 3b_2 \\ 0 & 0 & 0 & 0 & -15b_4 + 15b_1 \end{pmatrix}$$

   (b) Find an orthogonal basis for $Range(Q)$.

2. Let $v_1, v_2, \ldots v_7$ be the columns of the matrix A. A and the results of row reducing $(A \mid \theta)$ towards an echleon form and $(A \mid b)$ to echelon form are given.

$$A = \begin{pmatrix} 1 & 2 & 4 & 2 & 2 & 1 & 9 \\ 2 & 1 & 3 & 5 & 1 & -9 & 7 \\ 5 & 3 & 2 & 6 & 3 & -8 & 13 \\ 2 & 1 & 3 & 5 & 3 & -11 & 9 \\ 3 & 2 & 1 & 3 & 1 & -2 & 7 \\ 2 & 1 & 3 & 5 & 2 & -10 & 8 \end{pmatrix}. \qquad (A \mid \theta) \to \begin{pmatrix} 3 & 0 & 2 & 8 & 0 & 19 & 5 & 0 \\ 0 & 3 & 5 & -1 & 3 & 11 & 11 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$(A|b) \to \begin{pmatrix} 1 & 0 & \frac{2}{3} & \frac{8}{3} & 0 & -\frac{19}{3} & \frac{5}{3} & \frac{1}{3}(-2b_2 + b_1) \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & 1 & \frac{11}{3} & \frac{11}{3} & \frac{1}{3}(b_2 - 2b_1) \\ 0 & 0 & 1 & 1 & 0 & -2 & 1 & -\frac{3}{19}b_3 + \frac{1}{19}b_1 + \frac{7}{19}b_2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & \frac{1}{2}(b_4 - b_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 38b_5 - 4b_1 - 9b_2 - 26b_3 + 19b_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 38b_6 - 19b_2 - 19b_4 \end{pmatrix}.$$

   (a) Find a basis of the $Span(v_1, v_2, \ldots, v_7)$ **from** the given set of vectors $v_1, v_2, \ldots, v_7$.

   (b) Decide whether the vectors y and z given below are in the range(A) ?

$$y = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad z = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

3. The matrix Q along with the echelon forms of $(Q|b)$ and $(Q|\theta)$ are given. Find a basis for $Null(Q)$.

$$Q = \begin{pmatrix} 2 & 3 & 3 & 1 & 0 & 7 \\ 3 & 1 & 8 & 2 & 4 & 4 \\ 1 & 2 & 1 & 0 & -1 & 4 \\ 3 & 1 & 8 & 1 & 3 & 3 \\ -2 & -1 & -5 & 1 & 0 & -1 \end{pmatrix}. \qquad (Q|\theta) \to \begin{pmatrix} 1 & 0 & 3 & -1 & 0 & -1 & 0 \\ 0 & 7 & -7 & -1 & -8 & 13 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(Q|b) \to \begin{pmatrix} 1 & 0 & 3 & -1 & 0 & -1 & 4b_3 - 3b_1 + b_2 \\ 0 & 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{8}{7} & \frac{13}{7} & \frac{1}{7}(2b_2 - 3b_1) \\ 0 & 0 & 0 & 1 & 1 & 1 & \frac{1}{3}(-7b_3 + 5b_1 - b_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 3b_4 - 4b_2 - 7b_3 + 5b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3b_5 - 11b_1 + 4b_2 + 16b_3 \end{pmatrix}$$

4. The vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -2 \\ 1 \\ -2 \\ -2 \end{pmatrix}.$$

are a basis for a sub-space W of R^4. Find an orthogonal basis for W.

5. (a) Find a basis for $Null(M)$, where $M = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 4 & 5 & -1 & 6 \end{pmatrix}$.

   (b) The vectors $v = (-3, 1, 0, 1, 1)^T$ and $w = (3, -2, 0, 1, 0)^T$ are in $Null(M)$. Find vectors a, b, c, d, none of which are scalar multiples of v or w, such that the following conditions are true, or explain in one or two sentences why it is impossible to find such a vector.

   - $\{v, w, a\}$ is a basis for $Null(M)$.
   - $\{v, w, a, b\}$ is a basis for $Null(M)$.
   - $\{v, w, c\}$ is a linearly independent set of vectors that is not a basis for $Null(M)$.
   - $\{v, w, d\}$ is a linearly dependent set of vectors.

6. Let

$$A_{3\times 3} = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 4 & 8 \\ -1 & 0 & -2 \end{bmatrix}.$$

   (a) Decide whether the column vector's of the matrix $A_{3\times 3}$ are linearly independent or not.

   (b) If the columns A_1, A_2, A_3 of A are linearly dependent, express A_1 as a linear combination of A_2 and A_3. If they are independent, comment on the solution set for $Ax = b$, where $b_{3\times 1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

7. Solve the following questions, giving reasons for your answer. Please write in complete sentences. Feel free to ask me for extra sheets if you are short of space.

   (a) Let $A_{n\times n}$ be a matrix. Is $A + A^T$ symmetric or not ?

   (b) Let $A_{3\times 3} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Find A^{999}, A^{1000}.

   (c) You have 34×34 singular matrix A. Let b be a 34×1 vector. Describe all the possibilities for the solution set of the 34×34 system $Ax = b$.

   (d) If $A^3_{300\times 300} = \theta_{300\times 300}$ (zero matrix), can $A_{300\times 300}$ be non-singular ?

   (e) Suppose $C_{638\times 953}$ is a matrix whose echelon form has 308 **zero rows**. Find the Nullity(C^T).

   (f) Let W be a 3-dimensional subspace of R^5. We are given that

   $W = \text{span}\{\begin{pmatrix} 1 \\ -5 \\ \frac{17}{5} \\ 365 \\ \sqrt{23} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ \sqrt{17} \\ -8 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 0 \\ -9 \end{pmatrix}\}$. Find a linearly independent spanning set for W.

   (g) Let $W = \{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 3x_1 + 4x_2 - 6x_3 = 0\}$. Give an example of a non-zero vector that is orthogonal to all elements of W