For $f : \mathbb{R} \to \mathbb{R}$ and $x \neq y$, let
$$f[x,y] = \frac{f(x) - f(y)}{x - y}.$$

Convergence of Secant Method: *Let $f : \mathbb{R} \to \mathbb{R}$ and f be twice differentiable with continuous second derivative. Suppose*

- *there is a $c \in \mathbb{R}$ such that $f(c) = 0$ and $f'(c) \neq 0$.*

Then there is an $\eta > 0$ such that if $x_0, x_1 \in (c - \eta, c + \eta)$ then

(a) *for all $k \geq 1$,*
$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f[x_{k-1}, x_{k-2}]}$$
is a well defined sequence of real numbers with $x_k \in (c - \eta, c + \eta)$;

(b) *there is a $M > 0$ such that*
$$| x_k - c | \leq M | x_{k-1} - c || x_{k-2} - c |$$
for all $k \geq 2$;

(c) *$x_k \to c$ as $k \to \infty$; and*

(d) *there is a $C_1 > 0$ such that*
$$| x_k - c | \leq C_1 | x_{k-1} - c |^{\frac{1+\sqrt{5}}{2}}$$
for all $k \geq 1$.

Proof: As f is twice differentiable this implies that the derivative of f is continuous. As $f'(c) \neq 0$ there is a $0 < \delta < 1$, such that
$$f'(x) \neq 0 \text{ for all } x \in [c - \delta, c + \delta]. \qquad \text{(why ?)}$$
and hence
$$f[x,y] \neq 0 \text{ for all } x, y, \in [c - \delta, c + \delta] \text{ with } x \neq y. \qquad \text{(why ?)}$$

Fix
$$M = 1 + \sup_{z, x \neq y \in [c-\delta, c+\delta]} \left| \frac{f''(z)}{2f[x,y]} \right|, \qquad \eta = \frac{\delta}{M}, \qquad \text{and} \qquad x_0, x_1 \in (c - \eta, c + \eta).$$
Note that $1 \leq M < \infty$ (why ?) and hence $0 < \eta \leq \delta$.

Proof of (a) and (b): We shall proceed by induction.

- **Case $\mathbf{n = 2}$:** As $x_0, x_1 \in (c - \eta, c + \eta)$ and by definition of η, this implies that $f[x_0, x_1] \neq 0$ and
$$x_2 = x_1 - \frac{f(x_1)}{f[x_0, x_1]} \tag{7}$$
is well defined. Let $p : \mathbb{R} \to \mathbb{R}$ be given by
$$p(x) = f(x_1) + f[x_0, x_1](x - x_1).$$
Now,
$$f(x) - p(x) = f(x) - [f(x_1) + f[x_0, x_1](x - x_1)] = [f[x, x_1] - f[x_0, x_1]]\,(x - x_1) = f[x, x_1, x_0](x - x_0)(x - x_1).$$
Now there is a $\xi \in$ the smallest interval containing $\{x, x_1, x_0\}$, such that $f[x, x_1, x_0] = \frac{1}{2} f''(\xi)$ (See Worksheet 28-1-2020). Using this and the above we have
$$f(x) - p(x) = \frac{1}{2} f''(\xi)(x - x_0)(x - x_1)$$
As $f(c) = 0$, $p(c) = f(x_1) + f[x_0, x_1](c - x_1)$ we obtain
$$0 = f(x_1) + f[x_0, x_1](c - x_1) + \frac{1}{2} f''(\xi)(c - x_0)(c - x_1)$$

Using the fact from (7) that $0 = f(x_1) + f[x_0, x_1](x_2 - x_1)$, in the above and rearranging the terms we obtain

$$c - x_2 = \frac{1}{2}\frac{f''(\xi)}{f[x_0, x_1]}(c - x_1)(c - x_0).$$

Now, $\xi, x_0, x_1 \in (c - \eta, c + \eta) \subseteq [c - \delta, c + \delta]$. Using the definition of M, the above implies

$$| c - x_2 | \leq M | c - x_0 | | c - x_1 | . \quad (8)$$

and

$$| c - x_2 | < M\eta^2 = \delta\eta < \eta. \quad (9)$$

From (7), (8), (9) we have verified the case $n = 2$.

- **Fix $\mathbf{k \geq 2}$ and asssume for $\mathbf{2 \leq n \leq k}$:**

$x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f[x_{n-1}, x_n]}$ is well defined, $| x_n - c | \leq M | x_{n-1} - c | | x_{n-2} - c |$ with $x_n \in (c - \eta, c + \eta)$.

- **Case $\mathbf{n = k + 1}$:** From the induction hypothesis, we have that Now, $x_k, x_{k-1} \in (c - \eta, c + \eta)$. This implies that

$$x_{k+1} = x_k - \frac{f(x_k)}{f[x_{k-1}, x_k]} \quad (10)$$

is well defined.
Let $p : \mathbb{R} \to \mathbb{R}$ be given by

$$p(x) = f(x_k) + f[x_{k-1}, x_k](x - x_k).$$

Now,

$$\begin{aligned} f(x) - p(x) &= f(x) - [f(x_k) + f[x_{k-1}, x_k](x - x_k)] = [f[x, x_k] - f[x_{k-1}, x_k]]\,(x - x_k) \\ &= f[x, x_k, x_{k-1}](x - x_{k-1})(x - x_k) \end{aligned}$$

Now there is a $\xi_k \in$ the smallest interval containing $\{x, x_k, x_{k-1}\}$, such that $f[x, x_k, x_{k-1}] = \frac{1}{2}f''(\xi_k)$ (See Worksheet 28-1-2020). Using this and the above we have

$$f(x) - p(x) = \frac{1}{2}f''(\xi_k)(x - x_{k-1})(x - x_k)$$

As $f(c) = 0$, $p(c) = f(x_k) + f[x_{k-1}, x_k](c - x_k)$ we obtain

$$0 = f(x_k) + f[x_{k-1}, x_k](c - x_k) + \frac{1}{2}f''(\xi_k)(c - x_{k-1})(c - x_k)$$

Using the fact from (10) that, $0 = f(x_k) + f[x_{k-1}, x_k](x_{k+1} - x_k)$, in the above and rearranging terms we obtain

$$c - x_{k+1} = \frac{1}{2}\frac{f''(\xi_k)}{f[x_{k-1}, x_k]}(c - x_k)(c - x_{k-1}). \quad (11)$$

Now, $\xi_k, x_{k-1}, x_k \in (c - \eta, c + \eta) \subseteq [c - \delta, c + \delta]$. Using the definition of M, the above implies

$$| c - x_{k+1} | \leq M | c - x_{k-1} | | c - x_k | . \quad (12)$$

and

$$| c - x_{k+1} | < M\eta^2 = \delta\eta < \eta. \quad (13)$$

From (10), (12), (13) we have verified the case $n = k + 1$.

So by induction we have proved (a) and (b). □

Proof of (c): By part (a) and (b), we first note that for $n \geq 1$,

$$| x_n - c | \leq \frac{\eta}{M} | x_{n-1} - c | .$$

By an inductive argument it is easily seen we have for all $n \geq 1$,

$$| x_n - c | \leq \frac{\eta^n}{M^n} | x_0 - c | < \frac{\eta^{n+1}}{M^n} = \eta\delta^n < \delta^n.$$

Let $b > 0$ such that $\delta = \frac{1}{1+b}$. By induction we have that for all $n \geq 1$, $\delta^n < \frac{1}{nb}$. Let $\epsilon > 0$ be given. Let $N = \frac{1}{b\epsilon}$. For all $n \geq N$ we have that

$$| x_n - c | \leq \delta^n < \frac{1}{Nb} < \epsilon.$$

As $\epsilon > 0$ was arbitrary we have that $x_n \to c$ as $n \to \infty$. We have thus shown (c). □

Proof of (d): We shall assume without loss of generality that $x_n \neq c$ for all $n \geq 1$ and we will use the fact done in (11) in part (a) that for $k \geq 2$ there is $\xi_k \in$ the smallest interval containing $\{c, x_k, x_{k-1}\}$,

$$\frac{\mid c - x_{k+1} \mid}{\mid c - x_k \mid\mid c - x_{k-1} \mid} = \frac{1}{2} \mid \frac{f''(\xi_k)}{f[x_{k-1}, x_k]} \mid . \tag{14}$$

Now, $\xi_k, x_{k-1}, x_k \in (c - \eta, c + \eta) \subseteq [c - \delta, c + \delta]$. Us Let

$$p = \frac{1 + \sqrt{5}}{2} \text{ and so by our choice of } p \text{ we have } p^2 - p - 1 = 0$$

For $n \geq 1$, let

$$z_n = \frac{\mid x_n - c \mid}{\mid x_{n-1} - c \mid^p}.$$

Now, by our choice of p we have for $n \geq 2$

$$\begin{aligned}
z_n(z_{n-1})^{\frac{1}{p}} &= \frac{\mid x_n - c \mid}{\mid x_{n-1} - c \mid^p} \frac{\mid x_{n-1} - c \mid^{\frac{1}{p}}}{\mid x_{n-2} - c \mid} \\
&= \frac{\mid x_n - c \mid}{\mid x_{n-1} - c \mid\mid x_{n-2} - c \mid} \mid x_{n-1} - c \mid^{\frac{1}{p} - p + 1} \\
&= \frac{\mid x_n - c \mid}{\mid x_{n-1} - c \mid\mid x_{n-2} - c \mid} \mid x_{n-1} - c \mid^{\frac{1 - p^2 + p}{p}} \\
&= \frac{\mid x_n - c \mid}{\mid x_{n-1} - c \mid\mid x_{n-2} - c \mid}
\end{aligned}$$

Now using (14), there is a $\xi_n \in$ the smallest interval containing $\{c, x_n, x_{n-1}\}$

$$z_n(z_{n-1})^{\frac{1}{p}} = \frac{1}{2} \frac{f''(\xi_n)}{f[x_{n-1}, x_n]}$$

If $S_n = \frac{1}{2} \frac{f''(\xi_n)}{f[x_{n-1}, x_n]}$ then we have inductively for $n \geq 2$,

$$z_n = S_n(z_{n-1})^{-\frac{1}{p}} = \prod_{k=0}^{n} (S_{n-k})^{\frac{(-1)^k}{p^k}} (z_0)^{\frac{(-1)^n}{p^n}}$$

Now by assumptions on f we have that there is a $m_1 > 0, M_1 > 0$ such that

$$m_1 < S_n < M_1.$$

Hence there is a $C > 0$

$$z_n < C^{\sum_{k=0}^{n} \frac{1}{p^k}} (\alpha_0)^{\frac{1}{p^n}},$$

with $\alpha_0 = \max\{z_0, \frac{1}{z_0}\}$. As $p > 1$, we have that there is a $C_1 > 0$ such that $z_n < C_1$. $\square$