Convergence of Newton Raphson Method: *Let $f : \mathbb{R} \to \mathbb{R}$ and f be twice differentiable with continuous second derivative. Suppose*

- *there is a $c \in \mathbb{R}$ such that $f(c) = 0$ and $f'(c) \neq 0$.*

Then there is an $\eta > 0$ such that if $x_0 \in (c - \eta, c + \eta)$ then

(a) for all $k \geq 1$,

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$$

is a well defined sequence of real numbers;

(b) there is a $M > 0$ such that

$$\mid x_k - c \mid \leq M(x_{k-1} - c)^2$$

for all $k \geq 1$; and

(c) $x_k \to c$ as $k \to \infty$.

Proof As f is twice differentiable this implies that the derivative of f is continuous. As $f'(c) \neq 0$ there is a $0 < \delta < 1$, such that

$$f'(x) \neq 0 \text{ for all } x \in [c - \delta, c + \delta]. \qquad \text{(why ?)}$$

Fix

$$M = 1 + \sup_{x,y \in [c-\delta, c+\delta]} \mid \frac{f''(y)}{2f'(x)} \mid, \qquad \eta = \frac{\delta}{M}, \qquad \text{and} \qquad x_0 \in (c - \eta, c + \eta).$$

Note that $1 \leq M < \infty$ (why ?) and hence $0 < \eta \leq \delta$.

Proof of (a) and (b): We shall proceed by induction.

- **Case n = 1:** As $x_0 \in (c - \eta, c + \eta)$ and by definition of η, this implies that $f'(x_0) \neq 0$ and

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{1}$$

is well defined. By Taylor's Theorem there exists ξ_0 between x_0 and c such that

$$f(c) = f(x_0) + (c - x_0)f'(x_0) + \frac{(c - x_0)^2}{2} f''(\xi_0). \tag{2}$$

Now $f(c) = 0$ and from (1) we have that $f(x_0) = (x_0 - x_1)f'(x_0)$. Substituting these in (2) we have

$$0 = (x_0 - x_1)f'(x_0) + (c - x_0)f'(x_0) + \frac{(c - x_0)^2}{2} f''(\xi_0)$$

This implies that

$$x_1 - c = \frac{f''(\xi_0)}{2f'(x_0)}(x_0 - c)^2.$$

Note that $\xi_0, x_0 \in (c - \eta, c + \eta) \subseteq [c - \delta, c + \delta]$. Using the definition of M, the above implies

$$\mid x_1 - c \mid \leq M(x_0 - c)^2. \tag{3}$$

From (1) and (3) we have verified the case $n = 1$.

- **Fix $k \geq 1$ and asssume for $1 \leq n \leq k$:**

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \text{ is well defined and } \mid x_n - c \mid \leq M(x_{n-1} - c)^2$$

- **Case $\mathbf{n = k+1}$:** From the induction hypothesis and as $x_0 \in (c-\eta, c+\eta), 0 < \delta < 1$,

$$| x_k - c | \leq M(x_{k-1} - c)^2 \leq \frac{1}{M}\left[M(x_0 - c)\right]^{2^k} < \frac{1}{M}\left[M\eta\right]^{2^k} = \eta\delta^{2^k - 1} < \eta.$$

This implies that $f'(x_k) \neq 0$ and

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{4}$$

is well defined. By Taylor's Theorem there exists ξ_k between x_k and c such that

$$f(c) = f(x_k) + (c - x_k)f'(x_k) + \frac{(c - x_k)^2}{2} f''(\xi_k). \tag{5}$$

Now $f(c) = 0$ and from (4) we have that $f(x_k) = (x_k - x_{k+1})f'(x_k)$. Substituting these in (5) we have

$$0 = (x_k - x_{k+1})f'(x_k) + (c - x_k)f'(x_k) + \frac{(c - x_k)^2}{2} f''(\xi_k)$$

This implies that

$$x_{k+1} - c = \frac{f''(\xi_k)}{2f'(x_k)}(x_k - c)^2.$$

Note that $\xi_k, x_k \in (c-\eta, c+\eta) \subseteq [c-\delta, c+\delta]$. Using the definition of M, the above implies

$$| x_{k+1} - c | \leq M(x_k - c)^2. \tag{6}$$

From (4) and (6) we have proven the case $n = k+1$.

So by induction we have proved (a) and(b). □

Proof of (c): By part (b) and inductively we have for all $n \geq 1$,

$$| x_n - c | \leq \frac{1}{M}\left[M(x_0 - c)\right]^{2^n} \leq \frac{1}{M}\delta^{2^n} \leq \delta^{2^n}.$$

Let $b > 0$ such that $\delta = \frac{1}{1+b}$. So for all $n \geq 1$,

$$\delta^{2^n} = \frac{1}{(1+b)^{2^n}} < \frac{1}{(1+b)^n} < \frac{1}{nb}.$$

Let $\epsilon > 0$ be given. Let $N = \frac{1}{b\epsilon}$. For all $n \geq N$ we have that

$$| x_n - c | \leq \frac{1}{M}\delta^{2^n} \leq \delta^{2^n} < \frac{1}{Nb} < \epsilon.$$

As $\epsilon > 0$ was arbitrary we have that $x_n \to c$ as $n \to \infty$. We have thus shown (c). □

Since we have proved (a), (b) and (c) our proof is complete. □