Ordinary differential Equations (ODE's)

- Some examples from physics.
- Class XII level - a sparse introduction.
- A separate course on this subject is coming up in 3rd year.
- Illustrate Mathematical set up
  - briefly mention various technical aspects.
- Goal :- on finding "reasonable" good solutions.

XII - class view :-

$$\boxed{\cdot \quad \frac{dx}{dt} = x \qquad x(0) = 5}$$

Find x.

Solution:-

$$\Rightarrow \quad \frac{dx}{x} = dt$$

integrate both sides

$$\ln|x| + c_1 = t + c_2$$

$$\Rightarrow \quad x(t) = c_3 e^t$$

$$x(0) = 5 \quad \Rightarrow \quad \boxed{x(t) = 5e^t}$$

$\square$

<u>Definition</u> :- let $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous $T>0$, $\alpha \in \mathbb{R}$ is given. Then $x : [0,T] \longrightarrow \mathbb{R}$ is said to solve

$$[\text{ODE}] \longrightarrow \left.\begin{aligned} \frac{dx}{dt} &= f(t,x) \\ x(0) &= \alpha \end{aligned}\right\} \quad (1)$$

if

$$\left.\begin{aligned} x(t) = \alpha + \int_0^t f(s, x(s))\, ds \\ \text{for} \quad t \in [0,T]. \end{aligned}\right\} \quad (2)$$

Either (1) or (2) is reffered to as Initial value problem that solves the given ordinary differential equation.

Initial value Problem : Examples

Example 1 :- $T = 1$ $\quad f(t,x) = x+1$

$$\boxed{\begin{cases} \dfrac{dx}{dt} = x+1 \\ x(0) = 0 \end{cases}} \text{ — (IVP)}$$

Solution :- $x(t) = e^t - 1$ $\quad x : [0,T] \to \mathbb{R}$

is a solution. This is because

- $\dfrac{dx}{dt} = e^t = x+1$
- & $x(0) = e^0 - 1 = 0$.

One can actually show that there are no other solutions.

Characteristic of IVP :. There exists a unique solution to it

$\square$

Example 2 :- $T=1$.

$$f(t,x) = \begin{cases} \sqrt{x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

Initial value Problem :

(II) $\equiv$ $\dfrac{dx}{dt} = \sqrt{x}$; $x(t) = \int_0^t \sqrt{x(s)}\, ds$

$x(0) = 0$; $t \in [0,1]$

Solution 1 :: (*) $\begin{cases} x : [0,1] \to \mathbb{R} \\ x(t) = 0 \quad \forall\, t \in [0,1] \end{cases}$

clearly $x(0) = 0$

$0 < t < 1$, $0 = x(t)$

& $\int_0^t x(s)\, ds = \int_0^t 0\, ds = 0$

$\Rightarrow$ $x(t) = \int_0^t x(s)\, ds$.

x - given by (*) is a solution to IVP (II).

Solution 2 :-

$$x : [0,1] \to \mathbb{R}$$

$$x(t) = \frac{1}{4} t^2 \qquad t \in [0,1]$$

clearly $x(0) = 0$

$$\frac{dx}{dt} = \frac{1}{4} \cdot 2t = \frac{1}{2} t$$

$$= \sqrt{x} \qquad , t > 0$$

Fundamental Theorem of Calculus

$$x(t) = x(0) + \int_0^t \sqrt{x(s)}\, ds$$

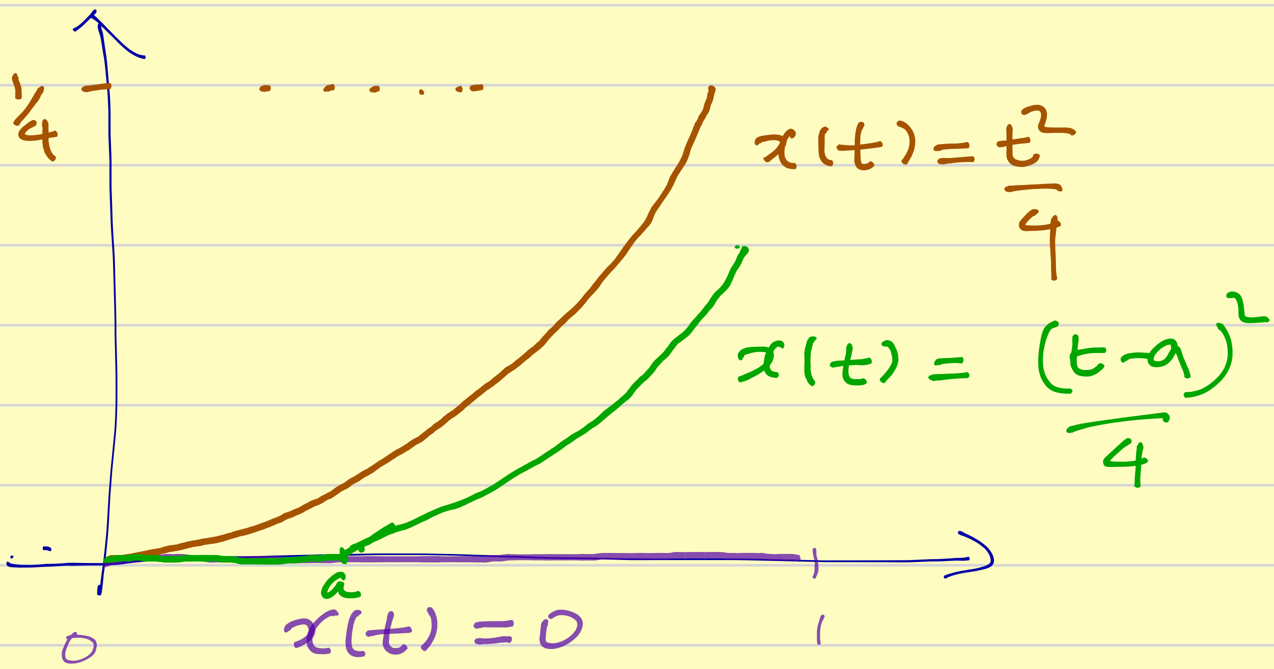
x — solves IVP (II).

There are two solutions IVP (II)
Infact there are infinitely many.

Other solutions :-
Let $0 < a < 1$. $x : [0,1] \to \mathbb{R}$

$$x(t) = \begin{cases} 0 & t \leq a \\ \frac{1}{4}(t-a)^2 & a < t < 1 \end{cases}$$

$$\frac{dx}{dt} = \sqrt{x(t)}, \quad x(0) = 0$$



- Infinitely many solutions.

The theory starts with the following result.

Theorem :- $T > 0$. $a \in \mathbb{R}$. $f : [0,T] \times \mathbb{R} \to \mathbb{R}$

(i) f is continous on $[0,T] \times \mathbb{R}$

(ii) $\frac{\partial f}{\partial t}(\cdot)$ and $\frac{\partial f}{\partial x}(\cdot)$ are bounded in $[0,T) \times \mathbb{R}$

Then there is a unique solution $x : [0,T) \to \mathbb{R}$ that solves the

(IVP) $$x(t) = a + \int_0^t f(s, x(s))\,ds$$

- Example 1 "can be applied" to Theorem to provide unique solution

- Theorem Does not apply to Example 2.

$\therefore\quad f(t,x) = \sqrt{x}$

$$\frac{\partial f(t,x)}{\partial x} = \frac{1}{2} x^{-1/2} \text{ is not}$$

bounded in $[0,1] \times [0,\infty)$

x ———————— x Goal of CS II x ————

$T>0 \quad x : [0,T] \to \mathbb{R} \qquad f : [0,T] \times \mathbb{R} \to \mathbb{R}$

$$x(t) = x(0) + \int_0^T f(s,x(s))\,ds$$

- Find or Construct a "suitable" x

we already know many Numerical Quadratures that allow us to provide "good" approximations to integrals.

Take $h>0$ - small. $t, t+h \in [0,T]$

$$x(t+h) - x(t) = \underbrace{\int_t^{t+h} f(s,x(s))\,ds}_{}$$

$h>0$ many methods to

Small : Perform numerical Quadrature

Rectangle Quadrature :-

$$(*)\quad x(t+h) - x(t) = \underbrace{f(t, x(t))\,h}_{\text{implicit formula}} + \underbrace{E(h)}_{\text{ERROR}}$$

Divide $[0, T]$:-

$0 \quad h \quad 2h \quad 3h \quad \cdots\cdots \quad T$

$x(h) = \cdots + E_1(h)$ Using $(*)$

$x(2h) = \cdots + E_2(h)$ $(*)$ again

$\vdots \qquad \vdots$

$x(T) = \cdots + E_n(h)$

linearly interpolate using $x(kh), x((k+1)h)$ in $[kh, (k+1)h]$.

Next class :-

- Trapezoid Rule
- Midpoint Rule

} Euler method

- ..... other discrete procedures } Runge-Kutta method

Vector-fields : [Discussion informal]

(IVP) - $\underbrace{\frac{dx}{dt} = f(t,x)}_{\text{understand it via picture}}$, $\underbrace{x(0) = \alpha}_{\text{stability}}$

stability ||| (Sensitivity to α.)

- slope of $x(\cdot)$ at (t_0, x_0) is given by $f(t_0, x_0)$.

$\approx$ imagine that a short line segment with slope $f(t_0, x_0)$ as a description of $x(\cdot)$ at (t_0, x_0)

- (t,x) plane and at "many" points we draw the above line segments

$\equiv$ illustration is what we call as vector-field