LU and Cholesky Decomposition

1. Let $A_{n\times n}$ be a non-singular matrix and $b_{n\times 1} \in \mathbb{R}^n$. Let g_n be the number of floating point operations required in performing Gaussian Elimination. Show that $g_n = O(n^3)$.

2. Let $A_{n\times n}$ be a square matrix of real numbers.

   (a) (*Review Exercise*) Show that there is a Permutation matrix $P_{n\times n}$ such that

   $$PA = LU$$

   where $L_{n\times n}$ is a lower triangular matrix and $U_{n\times n}$ is an upper triangular matrix.

   (b) Write an R-code called lupiv that takes as input $A_{n\times n}$ matrix and computes P, L and U.

3. Let $A_{n\times n}$ be a square matrix of real numbers be a symmetric positive definite matrix i.e. $x^T Ax > 0$ for $x_{n\times 1} \in \mathbb{R}^n$ with $x \neq 0$.

   (a) (Cholesky Decomposition) Show that there is a Lower triangular matrix $L_{n\times n}$ such that

   $$A = LL^T.$$

   Some authors (equivalently) say that there is an upperr triangular matrix $C_{n\times n}$ such that $A = C^T C$.

   (b) Write an R-code called cholesky that takes as input $A_{n\times n}$ matrix and computes the Cholesky Decomposition.

4. Using the R-code lupiv that you have written above and earlier codes on backsubstitution and forward substitution to solve for x when $Ax = b$ when

   $$A = \begin{bmatrix} 2 & 4 & -2 & -2 \\ 1 & 2 & 4 & -3 \\ -3 & -3 & 8 & -2 \\ -1 & 1 & 6 & -3 \end{bmatrix} \quad b = \begin{bmatrix} -4 \\ 5 \\ 7 \\ 7 \end{bmatrix}$$

5. Verify that

   $$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

   is a positive definite matrix. Using code of the cholesky that you have written above and earlier codes on backsubstitution and forward substitution to solve for x when $Ax = b$ when

   $$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \end{bmatrix}.$$