Gaussian Elimination & Pivoting

Solve: $Ux = b$ $\quad u_{ii} \neq 0$ and $u_{ij} = 0$, $i < j$

Backward substitution :-

$$x_n = \frac{b_n}{u_{nn}}$$

for $j = n-1, \dots, 1$
  $\alpha = b_j$
  for $k = j+1 \dots n$
    $\alpha = \alpha - u_{kj} x_k$ ✓
  end
  $x_j = \dfrac{\alpha}{u_{jj}}$
end

Solve :- $Ax = b$, $\underset{n\times n}{A}$ – non-singular $b_{n\times 1}$ $x_{n\times 1}$

- Consider $C = [A \vdots b]$
- Perform Elementary row operations and reduce C to Echelon form
  $[U \vdots \tilde{b}]$ U – upper triangular
- Backward substitution and solve
- Encounter zeros on diagonals & round-off error – (Scaled) Partial Pivoting

Gaussian Elimination with (Scaled) Partial Pivoting

A - non-singular

$$C = [A : b]$$

Partial Pivoting :- $S_i = \max_{1 \le j \le n} |C_{ij}|$ for $1 \le i \le n$ ← find order in each row

find (Smallest) i_p $\dfrac{|C_{i_p i}|}{S_{i_p}} = \max\left\{ \dfrac{|C_{ki}|}{S_k} : k \ge i \right\}$

Swap Row i_p and R_i ← Swapping

Algorithm :-

$C = [A : b]$

Compute $S_i = \max_{1 \le j \le n} |C_{ij}|$ $1 \le i \le n$.

for i in 1 to n-1

Partial Pivoting { find (smallest) i_p : $\dfrac{|C_{i_p i}|}{S_{i_p}} = \max\left\{ \dfrac{|C_{ki}|}{S_k} : k \ge i \right\}$ ✓ swap

Row i ⟺ Row_{i_p}.

for k in i+1 to n

for j in i to n+1

Elementary Row operations. { $C_{kj} = C_{kj} - \dfrac{C_{ki}}{C_{ii}} C_{ij}$ ✓ Sweep out rows

end

end

end

- Apply Back substitution Algorithm

Question :- How to handle Row swap in a program?

Row swap

$S = [s_1 \ldots s_n]$ and $l = [1 \ldots n]$ ← index vector

Step 1:

$j := \underset{\text{(smallest)}}{\operatorname{argmax}} \left\{ \frac{|a_{l_i,1}|}{s_{l_i}} : 1 \le i \le n \right\}$ (smallest)

- Swap $l_1 \Leftrightarrow l_j$ in l. ← Swap in l indexed
- Sweep out first column in C using $a_{l_1,1}$ in Row l_1.

Step 2.:

$j := \underset{\text{(smallest)}}{\operatorname{argmax}} \left\{ \frac{|a_{l_i,2}|}{s_{l_i}} : 2 \le i \le n \right\}$ 2nd row downward

- Swap $l_2 \Leftrightarrow l_j$ in l.
- Sweep out second column in C using $a_{l_2,2}$ in Row l_2

. . . .

Step k: $(1 \le k \le n)$

$j := \underset{\text{(smallest)}}{\operatorname{argmax}} \left\{ \frac{|a_{l_i,k}|}{s_{l_i}} : k \le i \le n \right\}$

- Swap $l_k \Leftrightarrow l_j$ in l.
- Sweep out k^{th} column in C using $a_{l_k,k}$ in Row l_k.

Step n+1:- Use backward substitution indexed by $l = [l_1 \ldots l_n]$

Example :-

$$\underset{4\times 4}{A} = \begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix} \qquad \underset{4\times 1}{b} = \begin{bmatrix} -19 \\ -34 \\ 16 \\ 26 \end{bmatrix}$$

$$S = [13 \quad 18 \quad 6 \quad 12], \qquad l = [1 \quad 2 \quad 3 \quad 4]$$

Step 1 :-

$$j := \underset{\text{(smallest)}}{\operatorname{argmax}} \left\{ \frac{|a_{l_i,1}|}{S_{l_i}} : 1 \le i \le 4 \right\}$$

$$= \underset{\text{(smallest)}}{\operatorname{argmax}} \left\{ \frac{3}{13}, \frac{6}{18}, \frac{6}{6}, \frac{12}{12} \right\}$$

$$= 3$$

- Swap $l_1 \leftrightarrow l_3$

$$\Rightarrow l = [3 \quad 2 \quad 1 \quad 4]$$

$$\begin{bmatrix} 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \\ 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \end{bmatrix}$$

← Row to sweep with

$$\begin{bmatrix} 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & 18 \\ 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \end{bmatrix}$$

Step2 :- $\Rightarrow l = [3 \ 2 \ 1 \ 4]$

$$j := \underset{(\text{smallest})}{\operatorname{argmax}} \left\{ \frac{|a_{l_i,2}|}{S_{l_i}} : \underbrace{2 \le i \le 4}_{} \right\}$$

2nd Column

2nd to 4th row

$$= \underset{(\text{smallest})}{\operatorname{argmax}} \left\{ \frac{2}{18}, \frac{12}{13}, \frac{4}{12} \right\}$$

$$= 3$$

Swap

$\Rightarrow l_2 \leftrightarrow l_3$

$\Rightarrow l = [3 \ 1 \ 2 \ 4.]$

$$\begin{bmatrix} 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & 18 \\ 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \end{bmatrix}$$

← Row to sweep with

$$\begin{bmatrix} 0 & -12 & 8 & 1 & -27 \\ 0 & 0 & 13/3 & -83/6 & -45/2 \\ 6 & -2 & 2 & 4 & 16 \\ 0 & 0 & -2/3 & \frac{5}{3} & 3 \end{bmatrix}$$

← Sweep l_3 row

* l_1 - don't touch.

← Sweep l_4 row

Step3 :- $l = [3 \ 1 \ 2 \ 4.]$

$$j := \underset{(\text{smallest})}{\operatorname{argmax}} \left\{ \frac{|a_{l_i,3}|}{S_{l_i}} : \underbrace{3 \le i \le 4}_{} \right\}$$

3rd Column

3rd & 4th row

$$= \underset{(\text{smallest})}{\operatorname{argmax}} \left\{ \frac{13/3}{18}, \frac{2/3}{12} \right\} = \operatorname{argmax}\{0.24, 0.06\}$$

$$= 3$$

- leave l unchanged

$\underline{l_3 \Leftrightarrow l_3}$

$$\begin{bmatrix} 0 & -12 & 8 & 1 & -27 \\ 0 & 0 & 13/3 & -83/6 & -45/2 \\ 6 & -2 & 2 & 4 & 16 \\ 0 & 0 & -2/3 & \frac{5}{3} & 3 \end{bmatrix}$$ ← Row to sweep with

$$\begin{bmatrix} 0 & -12 & 8 & 1 & -27 \\ 0 & 0 & 13/3 & -83/6 & -45/2 \\ 6 & -2 & 2 & 4 & 16 \\ 0 & 0 & 0 & -\frac{6}{13} & -\frac{6}{13} \end{bmatrix} \begin{matrix} \times \\ \times \\ \times \\ \end{matrix}$$ ⟵ Sweep l_4 row

Back-substitution $\qquad l = [3\ 1\ 2\ 4]$

✓ $l_4 = 4 \qquad -\frac{6}{13} x_4 = -\frac{6}{13} \implies \boxed{x_4 = 1}$ ✓

$l_3 = 2 \qquad \frac{13}{3} x_3 - \frac{83}{6} x_4 = -\frac{45}{2}$

$\implies \boxed{x_3 = -2}$ ✓

$l_2 = 1 \qquad -12 x_2 + 8 x_3 + x_4 = -27$

$\implies \boxed{x_2 = 1}$ ✓

$l_1 = 3 \qquad 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$

$\boxed{x_1 = 3}$ ✓

Exercise :- write a psuedo code for the above. ⟵ ?

Numerical stability :- Understand if numerical method can be trusted.

$Ax=b$ — we have discussed some methods for solving non-singular linear systems.

Effect of Round off error / measurements

A may be stored as $A+\delta A$

b may be stored as $b+\delta b$

$b \to b+\delta b$ Effect

let x be the solution : $Ax=b$

$x+\delta x$ be the solution : $A(x+\delta x)=b+\delta b$

where $A\delta x = \delta b$

Now let us consider a norm on space of vectors and matrices.

Say :-

$$\|D_{n\times n}\| = \left(\sum_{\substack{i=1\\j=1}}^{n} |d_{ij}|^2\right)^{\frac{1}{2}}$$

$$\|y_{n\times 1}\| = \left(\sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2}}$$

- $Ax = b \Rightarrow \|b\| = \|Ax\|$

[By standard norm properties]

$$\|b\| \leq \|A\| \, \|x\|$$

Say $x \neq 0$

$$\Rightarrow \boxed{\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}} \quad -(1)$$

- $A\,\delta x = \delta b$

$$\Rightarrow \delta x = A^{-1}\delta b$$

$$\Rightarrow \boxed{\|\delta x\| \leq \|A^{-1}\| \, \|\delta b\|} \quad -(2)$$

(1) and (2) gives

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \, \frac{\|\delta b\|}{\|b\|}$$

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\| \, \|A^{-1}\|}_{\text{Condition number} := K(A)} \, \frac{\|\delta b\|}{\|b\|}$$

Understanding(s) :-

- If $K(A)$ is of order 10^k then - expect to loose at least k-digits of precision in solving $Ax = b$

- large condition number of A $\Rightarrow$ $Ax = b$ is sensitive to perturbations in b

ill-conditioned - matrices

Example :- Hilbert Matrices

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

Find : $K(H) =$

$\det(H) =$