Question 2. Suppose that $n \geq 2$, $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ is such that, $f^{(k}(a) = 0$ for all $k \leq n-1$ and $f^{(n)}(a) \neq 0$. If $f^{(n)}(\cdot)$ is continuous at a then show that

(a) if n is even and $f^{(n)}(a) > 0$ then f has a local minimum at a,

(b) if n is even and $f^{(n)}(a) < 0$ then f has a local maximum at a,

(c) if n is odd then f has a point of inflection at a.

Solution: We are given: $n \geq 2$; $f : \mathbb{R} \to \mathbb{R}$; $a \in \mathbb{R}$ such that $f^{(n)}$ is continuous at a;

$$f^{(k)}(a) = 0, \forall\ k \leq n-1; \text{ and } f^{(n)}(a) \neq 0. \tag{1}$$

By Taylor's theorem, there exists $\delta > 0$ such that for any $x \in (a-\delta, a+\delta)$, there exists $c \equiv c(x,a)$ that lies between x and a such that

$$f(x) = f(a) + \sum_{k=1}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(c).$$

From our hypothesis,(1), this implies there exists $\delta > 0$ such that for any $x \in (a-\delta, a+\delta)$, there exists $c \equiv c(x,a)$ that lies between x and a such that

$$f(x) = f(a) + \frac{(x-a)^n}{n!} f^{(n)}(c). \tag{2}$$

Now $f^{(n)}$ is continuous at a. Let $\epsilon = \frac{|f^{(n)}(a)|}{2}$ there exists $0 < \delta_1 < \delta$ such that

$$\mid f^{(n)}(t) - f^{(n)}(a) \mid < \epsilon \qquad \forall\ t \in (a-\delta_1, a+\delta_1). \tag{3}$$

(a)Let n is even and $f^{(n)}(a) > 0$.
Let $x \in (a-\delta_1, a+\delta_1)$, let c be as in (2). This will imply $c \in (a-\delta_1, a+\delta_1)$. So from (3) we have

$$\mid f^{(n)}(c) - f^{(n)}(a) \mid < \frac{f^{(n)}(a)}{2} \qquad \Longrightarrow \qquad f^{(n)}(c) > \frac{f^{(n)}(a)}{2}$$

. Further $(x-a)^n > 0$, as n is even number. Thus, for any $x \in (a-\delta_1, a+\delta_1)$ using this and (2) we have

$$f(x) = f(a) + \frac{(x-a)^n}{n!} f^{(n)}(c) > f(a)$$

Therefore f has a local minimum at a. $\square$

(b)Let n is even and $f^{(n)}(a) < 0$.

Let $x \in (a-\delta_1, a+\delta_1)$, let c be as in (2). This will imply $c \in (a-\delta_1, a+\delta_1)$. So from (3) we have

$$| f^{(n)}(c) - f^{(n)}(a) |< \frac{-f^{(n)}(a)}{2} \qquad \Longrightarrow \qquad f^{(n)}(c) < \frac{f^{(n)}(a)}{2}$$

. Further $(x-a)^n > 0$, as n is even number. Thus, for any $x \in (a-\delta_1, a+\delta_1)$ using this and (2) we have

$$f(x) = f(a) + \frac{(x-a)^n}{n!} f^{(n)}(c) < f(a)$$

Therefore f has a local maximum at a. $\square$

(c) Let n be odd.

Note that $(x-a)^n > 0$ for $x > a$ and $(x-a)^n < 0$ for $x < a$. Let $f(n)(c) > 0$. Then using (2) $f(x) < f(a)$ for $x < a$ and $f(x) > f(a)$ for $x > a$. So a is neither a local maximum nor a local minimum of f. Thus a is an inflection point at f. $\square$

Question 3: For each of the following indicate whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$,$f(n) = \Theta(g(n))$,$f(n) = o(g(n))$

(a) $f(n) = 100n$ and $g(n) = n^{1.1}$

(g) $f(n) = \frac{2^n}{n!}$ and $g(n) = n^{-0.9n}$

Solution: We say that:

- $f(n) = O(g(n))$ if there exists $N_0 \in \mathbb{N}$ and $c > 0$ such that $f(n) \leq cg(n)$ for all $n \geq N_0$
- $f(n) = \Omega(g(n))$ if there exists $N_0 \in \mathbb{N}$ and $c > 0$ such that $f(n) \geq cg(n)$ for all $n \geq N_0$
- $f(n) = \Theta(g(n))$ if there exists $N_0 \in \mathbb{N}$ and $c_1, c_2 > 0$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq N_0$
- $f(n) = o(g(n))$ if for every $c > 0$ there exists N_0 such that $f(n) \leq cg(n)$ for all $n \geq N_0$

It is clear that if $f(n) = o(g(n))$. This immediately implies that $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$. Consequently, $f(n) \neq \Theta(g(n))$, Secondly, if $f(n) = \Theta(g(n))$ this immediately implies that $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$ and $f(n) \neq o(g(n))$.

(a) Let $n \geq 1$, $f(n) = 100n$ and $g(n) = n^{1.1}$. Then

$$\frac{f(n)}{g(n)} = \frac{100}{n^{0.1}}$$

Let $\epsilon > 0$ be given and $N_0 = \left\lceil \left(\frac{100}{\epsilon}\right)^{10} \right\rceil$. For all $n \geq N_0$,

$$0 \leq \frac{f(n)}{g(n)} < \epsilon.$$

As $\epsilon > 0$ we have $f(n) = o(g(n))$. $\square$

(g) $f(n) = \frac{2^n}{n!}$ and $g(n) = n^{-0.9n}$. We can verify using mathematical induction that for all $n \geq 3$.

$$n! \geq n^n 3^{-n}.$$

Then for all $n \geq 3$,

$$0 \leq \frac{f(n)}{g(n)} = \frac{2^n n^{0.9n}}{n!} < \left(\frac{6}{n^{0.1}}\right)^n.$$

Let $\epsilon > 0$ be given. For $n \geq 6^{11}$, there is a $0 < \delta < 1$

$$\frac{6}{n^{0.1}} < \delta.$$

Let $\delta = \frac{1}{1+b}$ for some $b > 0$. Then using Binomial expansion for all $n \geq 1$, $(1+b)^n \geq nb$. Therefore, for $n \geq 6^{11}$, we have

$$0 \leq \frac{f(n)}{g(n)} = \frac{2^n n^{0.9n}}{n!} < \delta^n = \frac{1}{(1+b)^n} \leq \frac{1}{nb}.$$

Let $N_0 = \frac{1}{b\epsilon}$. For all $n \geq N_0$

$$0 \leq \frac{f(n)}{g(n)} < \epsilon.$$

As $\epsilon > 0$ we have $f(n) = o(g(n))$. $\square$