$\underline{T > 0}$

$x : [0,T] \to \mathbb{R}$

$$\frac{dx}{dt} = f(t,x) \quad , \quad x(0) = \alpha$$

$f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

1st order O.D.E.'s

1 - equation.

Numerical Methods

linear system of O.D.E's

$x : [0,T] \to \mathbb{R}^d$

$$\frac{dx}{dt} := \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_d}{dt} \end{bmatrix}$$

$x_i : [0,T] \to \mathbb{R}$

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{d} a_{ij}\, x_j(t) \quad , \quad \begin{matrix} 1 \le i \le d \\ 1 \le j \le d \end{matrix}$$

$a_{ij} \in \mathbb{R}$

In short : $x(0) \in \mathbb{R}^d$

and $\dfrac{dx}{dt} = Ax$ $\qquad A = [a_{ij}]$

Guess of solution :-

$$\frac{dx}{dt} = Ax \quad , \quad x(0) = \alpha.$$

$$\Rightarrow \quad x(t) = \alpha e^{At}$$

what is e^{A} ?

Alternative approach

let v be an eigen vector of A

λ be the corresponding eigen value $\qquad Av = \lambda v$

Take $\boxed{x(t) = e^{\lambda t} v}$

$$\frac{dx}{dt} = \lambda e^{\lambda t} v$$

[Coordinate wise differentiation]

$$= e^{\lambda t} A v$$

$$\boxed{\frac{dx}{dt} = Ax}$$

let $\lambda_1, \ldots, \lambda_d$ are eigen values of A

$v_1, \ldots v_d$ are eigen vectors of A

$$x(t) = \sum_{i=1}^{d} \alpha_i e^{\lambda_i t} v_i$$

would be a solution to

$$\frac{dx}{dt} = Ax.$$

Determine $\alpha_i : 1 \leq i \leq d$ by restrictions on $x(0)$.

<u>Non-linear system</u> : $d = 2$.

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2)$$

$$\frac{dx_2}{dt} = f_2(t, x_1, x_2)$$

In vector notation :-

$$\frac{dx}{dt} = f(t, x)$$

$x : [0, T] \longrightarrow \mathbb{R}^2$

$f : [0, \infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Numerical Methods generalise easily

Runge-Kutta order 4 algorithm

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2)$$

$$\frac{dx_2}{dt} = f_2(t, x_1, x_2)$$

$(x_{11}, x_{21}), \ldots \qquad , (x_{1j}, x_{2j})$

$$K_{1,1} = f_1(t_j, x_{1j}, x_{2j})$$
$$K_{1,2} = f_2(t_j, x_{1j}, x_{2j})$$

$$K_{2,1} = f_1\left(t_j + \frac{h}{2}, x_{1j} + \frac{h}{2}k_{1,1}, x_{2j} + \frac{h}{2}k_{1,2}\right)$$
$$K_{2,2} = f_2\left(t_j + \frac{h}{2}, x_{1j} + \frac{h}{2}k_{1,1}, x_{2j} + \frac{h}{2}k_{1,2}\right)$$

$$K_{3,1} = f_1\left(t_j + \frac{h}{2}, x_{1j} + \frac{h}{2}k_{2,1}, x_{2j} + \frac{h}{2}k_{2,2}\right)$$

$$K_{3,2} = f_2\left(t_j + \frac{h}{2}, x_{1j} + \frac{h}{2}k_{2,1}, x_{2j} + \frac{h}{2}k_{2,2}\right)$$

$$K_{4,1} = f_1\left(t_j + h,\ x_{1j} + h k_{3,1},\ x_{2j} + h k_{3,2}\right)$$

$$K_{4,2} = f_2\left(t_j + h,\ x_{1j} + h k_{3,1},\ x_{2j} + h k_{3,2}\right)$$

$$x_{1,j+1} = x_{1,j} + h\left[\frac{k_{1,1}}{6} + \frac{k_{2,1}}{3} + \frac{k_{3,1}}{3} + \frac{k_{4,1}}{6}\right]$$

$$x_{2,j+1} = x_{2,j} + h\left[\frac{k_{1,2}}{6} + \frac{k_{2,2}}{3} + \frac{k_{3,2}}{3} + \frac{k_{4,2}}{6}\right]$$

Proceed inductively forward.

Examples :- [Predator - Prey]

P_1 - Prey population

P_2 - Predator population

$$\boxed{\begin{aligned} \frac{dP_1}{dt} &= aP_1 - \delta_1 P_1 P_2 \\ \frac{dP_2}{dt} &= \delta_2 P_1 P_2 - bP_2 \end{aligned}}$$ — non linear

<u>Example</u> : (Susceptible - infected Recovered)

S = number of susceptible individuals

I = number of infected individuals

R = number of recovered individuals

N - total number of individuals in population.

$$s(t) = \frac{S(t)}{N}, \quad i(t) = \frac{I(t)}{N}, \quad r(t) = \frac{R(t)}{N}$$

$$s(t) + i(t) + r(t) = 1 \quad - (*)$$

$$\frac{dS}{dt} = -b\,s\,I \quad - (1)$$

$$\frac{dI}{dt} = \cdots \quad ? \quad \cdots \quad (2)$$

$$\frac{dR}{dt} = k\,I \quad - (3)$$

From (*) : $\dfrac{ds}{dt} + \dfrac{di}{dt} + \dfrac{dr}{dt} = 0 \quad - (**)$

From (1)

(3)

(**) $+ \cdots \to$ (2)

$$\boxed{\begin{aligned} \frac{ds}{dt} &= -b\,s\,i \\ \frac{dr}{dt} &= k\,i \\ \frac{di}{dt} &= b\,s\,i - k\,i \end{aligned}}$$

SIR model
(mean-field)