Ordinary Differential Equations

① $\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(0) = \alpha \end{cases}$ - typically use this notation

$\Updownarrow$

② $\begin{cases} x : [0,T] \to \mathbb{R} \\ x(t) = \alpha + \int_0^t f(s, x(s))\,ds \end{cases}$

Q: How to find solution to ② given an $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$?

ANS:- . analytic methods – IIIrd year

. Basic Theorem : Existence & uniqueness (stated last time)

Examples of f

• Numerical approximations – in this class.

Numerical Methods: $x : [0,T] \to \mathbb{R}$

$$x(t) = x(0) + \int_0^t f(s, x(s))\, ds \quad -(1)$$

$[0,T]$ $\quad t_0 = 0$

$h = \frac{T-0}{n}, \; n \geq 1$

$t_i = t_{i-1} + h$

$i = 1, \ldots n$

$$x(t_i) = x(0) + \int_0^{t_i} f(s, x(s))\, ds$$

$$= x(0) + \int_0^{t_{i-1}} f(s, x(s))\, ds$$

$$+ \int_{t_{i-1}}^{t_i} f(s, x(s))\, ds$$

$$\boxed{x(t_i) = x(t_{i-1}) + \int_{t_{i-1}}^{t_i} f(s, x(s))\, ds \qquad i = 0, 1, \ldots n}$$

Idea 1 :-

(a) • - We have learnt many Numerical quadrature to approximate integrals.

(Trapezoid, rectangle, midpoint)

(b) - Use quadrature to compute inductively

$x(t_1)$, $x(t_2)$ ... , $x(t_n)$

(c) - $\{(t_i, x(t_i)) : i = 0, .., n\}$

use any of the interpolation techniques to construct x.

(d) - Error Analysis is available

- f, x - n, h

- vector field - notion discussed last class.

Formally speaking - "dashes" of line with slopes given by $f(s_i, y_i)$

Euler's Method :-

Assume :-

$x : [0,T] \to \mathbb{R}$ $\qquad f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

$$x(t) = x(0) + \int_0^t f(s, x(s))\,ds \quad \text{--- (1)}$$

$t \in [0,T]$

is twice differentiable.

(Taylor approximation)

$[0,T]$ $\qquad t_0 = 0$

$t_i = t_{i-1} + h$

$i = 1, \dots n$

$h = \frac{T-0}{n}, \; n \geq 1$

$\xi \in [t_i, t_{i-1})$

$$x(t_i) = x(t_{i-1}) + h\,x'(t_{i-1}) + \frac{h^2}{2} x''(\xi)$$

$$= x(t_{i-1}) + h\,f(t_{i-1}, x_{t_{i-1}}) + E(h)$$

Euler's Method :-

$$x(0) = x(0)$$

$$x(t_i) = x(t_{i-1}) + h\,f(t_{i-1}, x_{t_{i-1}})$$

$i = 1, \dots, n$

$SE(t_i, t_{i-1})$ — Use interpolation techniques

- Start — with linear interpolation to construct x.

$\underbrace{\text{Start — with linear interpolation to construct } x}_{\text{Error introduced}}$

Analysis of Euler's Method :-

$z : [0,T] \to \mathbb{R}$

$$z(t) = \alpha + \int_0^t f(s, z(s))\, ds$$

[Exact Solution]

$x : [0,T] \to \mathbb{R}$

$$x(t_i) = x(t_{i-1}) + h f(t_{i-1}, x_{t_{i-1}})$$

$i = 1, \dots n$

$$x(0) = \alpha$$

- $SE(t_{i-1}, t_i)$ — linear interpolation

This may happen :-

$$x(t_{i-1}) \neq z(t_{i-1})$$

Taylor's Theorem

$$z(t_i) - z(t_{i-1}) = h f(t_{i-1}, z_{t_{i-1}}) + \frac{h^2}{2} z''(\xi_i)$$

$$\Rightarrow \frac{z(t_i) - z(t_{i-1})}{h} - f(t_{i-1}, z_{t_{i-1}}) = \frac{h}{2} z''(\xi_i)$$

$$\equiv$$

$$\tau(h, t_i) = \boxed{\frac{h}{2} z''(\xi)}$$

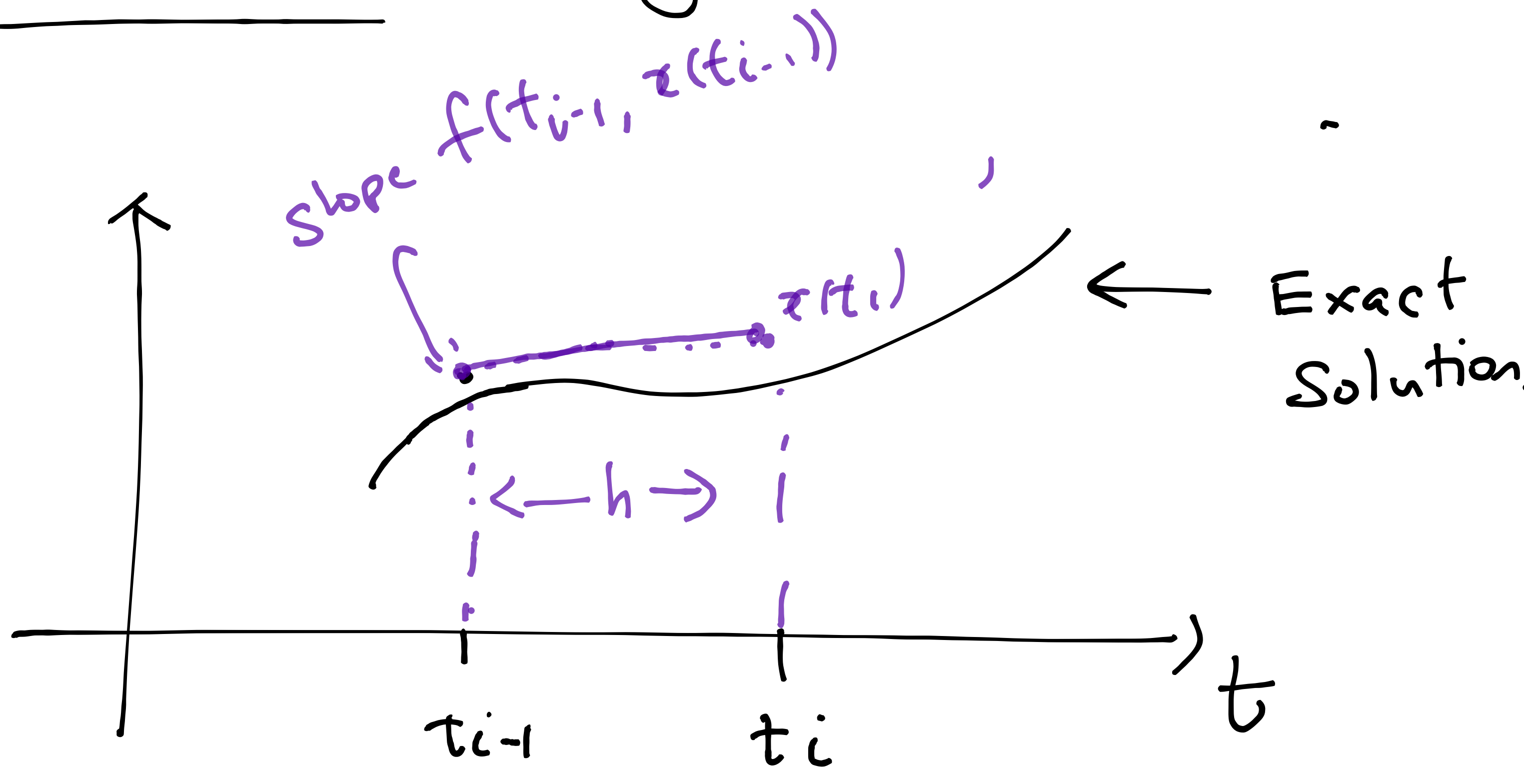
$$\frac{x(t_i) - x(t_{i-1})}{h} - f(t_{i-1}, x_{t_{i-1}})$$

$$\tau(h, t_i) \equiv \text{local discretization error (LDE)} \equiv \frac{h}{2} z''(\xi_i)$$

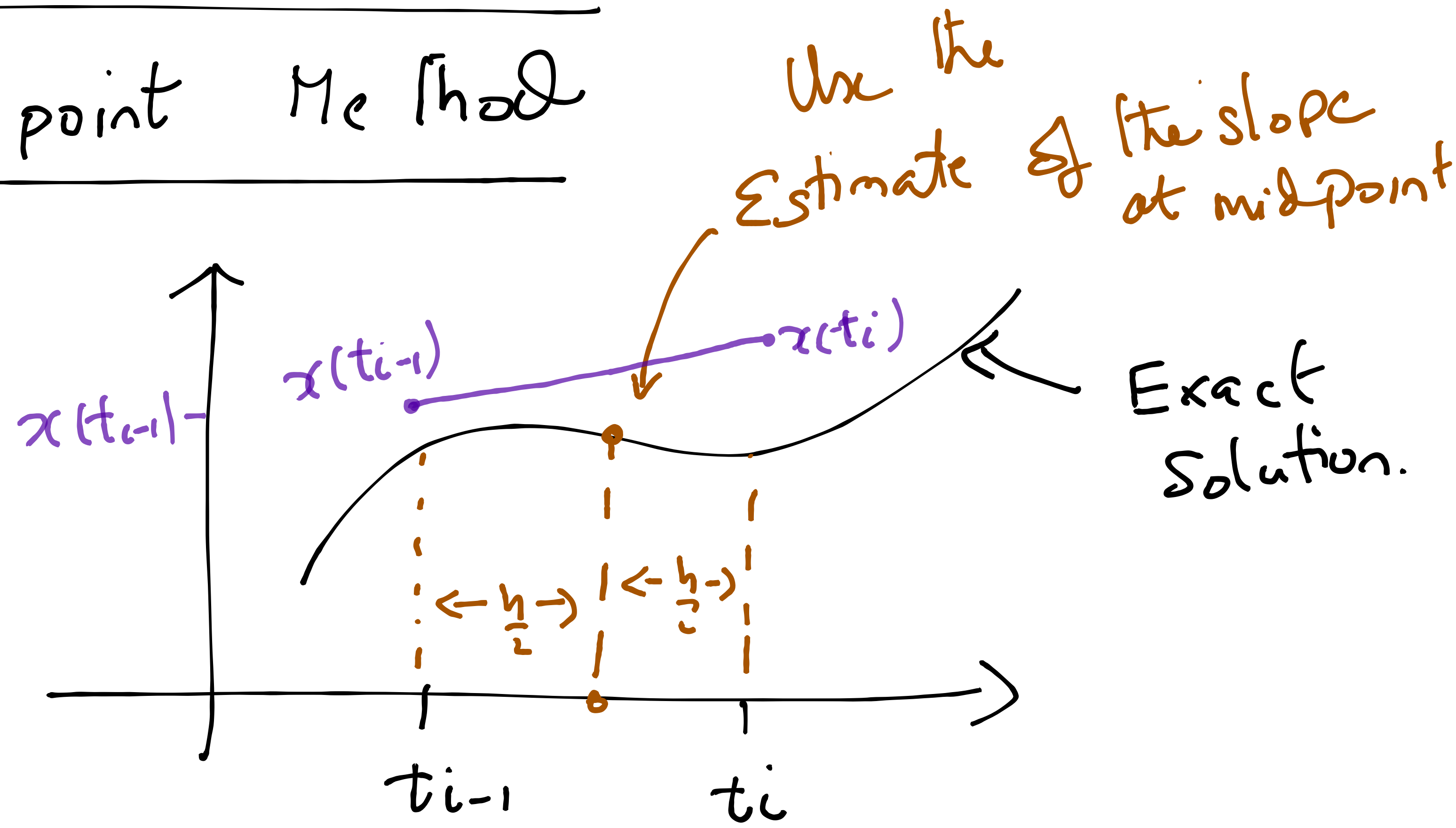
$z''(\cdot)$ bounded $\Rightarrow$

$$\text{LDE} = O(h)$$

Euler's Method (graphic)



Midpoint Method



$x(t_0), \ldots, x(t_{i-1})$ – we already have constructed

$$K_1 = f(t_{i-1}, x(t_{i-1}))$$

- Euler :- $x\left(\frac{t_{i-1}+t_i}{2}\right) = x(t_{i-1}) + \frac{h}{2} K_1$

Estimate of the slope at mid point

$$k_2 = f\left(\frac{t_{i-1}+t_i}{2}, x(t_{i-1}) + \frac{h}{2}k_1\right)$$

MIDPOINT RULE :-

$$\boxed{\begin{aligned} x(t_i) &= x(t_{i-1}) + h\,k_2 \qquad i = 1, \dots, n \\ x(t_0) &= x(0) \end{aligned}}$$

Fact :- LDE $= O(h^2)$.

- The above methods & their Error analysis rely on the Taylor approximation of x.

- One could also try Taylor approximation to $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

$$\underbrace{\text{approximation to } f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}}_{\text{Carl Runge - Kutta, Wilhelm}}$$

$$f(t+h, x+k) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial x}\right)^m f(t,x)$$

where $\displaystyle\sum_{j=0}^{m} {}^mC_j \, h^j k^{m-j} \frac{\partial^j}{\partial t^j} \frac{\partial^{m-j}}{\partial x^{m-j}} = \left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial x}\right)^m$

Upto 1^{st} order

$$f(t+h, x+k) = f(t,x) + \left(h\frac{\partial f}{\partial t} + k\frac{\partial f}{\partial x}\right) + \text{Error}$$

Runge-kutta Method of order 2 :-

$h>0$, $k>0$, (t,x) = Given

$$K_1 = h f(t,x)$$

$$K_2 = h f(t+\alpha, x+\beta K_1)$$

Set :- $x(t+h) = x(t) + \omega_1 K_1 + \omega_2 K_2$

$\Rightarrow$

$$x(t+h) = x(t) + w_1 h f(t, x(t)) + w_2 h f(t+\alpha, x(t)+\beta k_1) \quad - ③$$

Find : w_1, w_2, α, β

Ex :- Use Taylor series for f, x

$$\begin{cases} w_1 + w_2 = 1 \\ \alpha w_2 = \frac{1}{2} \\ \beta w_2 = \frac{1}{2} \end{cases} \Rightarrow \begin{array}{l} \alpha = 1 \\ \beta = 1 \\ w_1 = \frac{1}{2} \\ w_2 = \frac{1}{2} \end{array}$$

Runge-Kutta Method of order 2 :-

Single extension

$$x(t+h) = x(t) + \frac{1}{2}(K_1 + K_2)$$

Computing slope at 2 points

$$\begin{cases} K_1 = h f(t, x(t)) \\ K_2 = h f(t+h, x(t)+K_1) \end{cases}$$

$[0, T]$: $t_0 = 0$ $\quad h = \frac{T-0}{n}$

$$t_i = t_{i+1} + h$$

$$x(t_i) = x(t_{i-1}) + k_1 + k_2$$

$$k_1 = h f(t_{i-1}, x(t_{i-1}))$$

$$k_2 = h f(t_{i-1} + h, x(t_{i-1}) + k_1)$$

$$i = 1, \dots, n$$

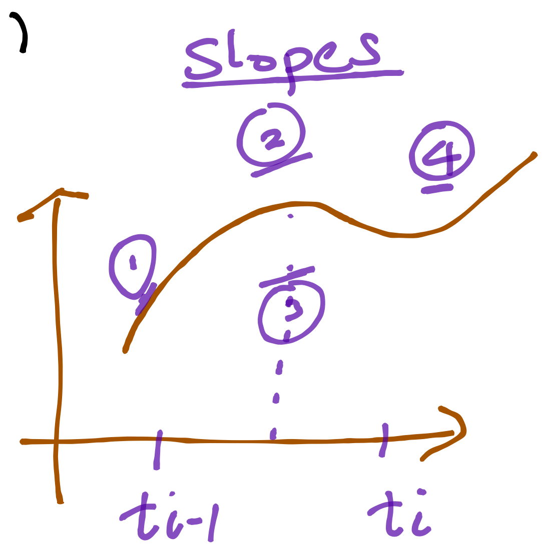
$$x(t_0) = x(0)$$

Fact :- LDE $\equiv O(h^3)$.

Runge - Kutta Method of order 4 :-

$x : [0,T] \to \mathbb{R}$, $h = \frac{T-0}{n}$, $t_0 = 0$

$t_i = t_{i-1} + h$ $\quad i = 1, \dots,$



For each $i = 1, \dots n$:

$$k_1 = f(t_{i-1}, x(t_{i-1}))$$

$$k_2 = f\left(t_{i-1} + \frac{h}{2}, x(t_{i-1}) + \frac{h}{2} k_1\right)$$

$$K_3 = f\left(t_{i-1} + \frac{h}{2},\ x(t_{i-1}) + \frac{h}{2} k_2\right)$$

$$K_4 = f\left(t_{i-1} + h,\ x(t_{i-1}) + h\,K_3\right)$$

$$x(t_i) = x(t_{i-1}) + h\left[\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}\right]$$

<u>Fact</u> :- $LDE \equiv O(h^4)$

<u>Extensions</u> :-

- Higher-order Runge Kutta
- <u>Understand</u> "h" :- step size

- Adaptive step size

Eg: – h_1 – start
– $h_2 = \frac{h_1}{2}$ or h_1

– User : - Error tolerance
- Try and determine appropriate step size.