(f) Let $n \to \infty$ and then $K \to \infty$ in (**??**) to conclude the result (using (**??**)).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space.. Let $\{X_n\}$ be a sequence of i.i.d. samples on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider $\bar{X}_n$, the empirical mean given by

$$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k.$$

Let for $u \in \mathbb{R}$,

$$s(u) = \sup_{n\geq 1} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \geq u)$$

and for $\lambda \in \mathbb{R}$

$$p(\lambda) = \log E[e^{\lambda X_1}].$$

We shall work in $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$.

In this worksheet we shall prove the following result on Large Deviation.

Cramérs Theorem: For all $x \in \mathbb{R}$, the sequence

$$\frac{1}{n} \log \mathbb{P}(\bar{X}_n \geq x)$$

converges in $[-\infty, 0]$ and

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \geq x) = \inf_{\lambda\geq 0} (\log E[e^{\lambda X_1}] - \lambda x)$$

1. Fix $m \geq 1$, for any $n \geq 1$ let $q_n, 0 \leq r_n \leq m$ be such that $n = q_n m + r_n$.

   (a) Show that

   $$\{\bar{X}_n \geq x\} \supset \bigcap_{k=0}^{q_n - 1} \left\{ \frac{1}{m} \sum_{i=mk+1}^{m(k+1)} X_i \geq x \right\} \bigcap_{i=mq_n+1}^{n} \{X_i \geq x\}$$

   and conclude that

   $$\mathbb{P}(\{\bar{X}_n \geq x\}) \geq \mathbb{P}(\bar{X}_m \geq x)^{q_n} \mathbb{P}(X_1 \geq x)^m.$$

   (b) Show that

   $$\frac{q_n}{n} \to \frac{1}{m}, \text{ as } n \to \infty.$$