1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space.. Let $\{X_n\}$ be a sequence of i.i.d. samples on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider $\bar{X}_n$, the empirical mean given by

$$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k.$$

Let for $u \in \mathbb{R}$,

$$s(u) = \sup_{n\geq 1} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \geq u)$$

and for $\lambda \in \mathbb{R}$

$$p(\lambda) = \log E[e^{\lambda X_1}].$$

We shall work in $\mathbb{R}\cup\{-\infty\}\cup\{\infty\}$. In this worksheet we will prove the result: For all $\lambda \geq 0$,

$$p(\lambda) = \sup_{u\in\mathbb{R}}(\lambda u + s(u)) \tag{1}$$

(a) Show (1) for $\lambda = 0$. i.e.

$$\sup_{u\in\mathbb{R}} s(u) = 0 = p(0).$$

(b) Show that

$$P(\{\bar{X}_n > u\}) \leq (E[e^{\lambda X_1}])^n e^{-n\lambda u}$$

for $\lambda \geq 0$ and conclude that

$$p(\lambda) \geq \sup_{u\in\mathbb{R}}(\lambda u + s(u)) \tag{2}$$

for all $\lambda > 0$

(c) Show that for $K > 0$ large enough and any $lambda > 0$

$$-\lambda K < \sup_{u\in R}(\lambda u + s(u)) \tag{3}$$

(d) Let $\lambda > 0$ and $K > 0$ prove each of the (in)equalities below:

$$\begin{aligned}
\log E[e^{\lambda X_1}\ 1(\mid X_1 \mid \leq K)] &\leq \frac{1}{n}\log E[e^{n\lambda\bar{X}_n}\ 1(\mid \bar{X}_n \mid\leq K)] \\
&= \frac{1}{n}\log E[\left(e^{-n\lambda K} + \int_{-K}^{\bar{X}_n} e^{n\lambda u} n\lambda du\right)\ 1(\mid \bar{X}_n \mid \leq K)] \\
&\leq \frac{1}{n}\log\left(e^{-n\lambda K} + \int_{-\infty}^{\infty} E\left[\ 1(-K \leq u \leq \bar{X}_n)\ 1(\mid \bar{X}_n \mid\leq K) e^{n\lambda u} n\lambda du\right]\right) \\
&\leq \frac{1}{n}\log\left(e^{-n\lambda K} + \int_{-K}^{K} E\left[e^{n(\lambda u + s(u))} n\lambda du\right]\right)
\end{aligned} \tag{4}$$

(e) Using (**??**) and (**??**) show that

$$\log E[e^{\lambda X_1}\ 1(\mid X_1 \mid\leq K)] \leq \frac{1}{n}\log(1 + 2Kn\lambda) + \sup_{u\in\mathbb{R}}(\lambda u + s(u)) \tag{5}$$