Due Date: 26th September 2022, 11am

1. Let $(\Omega, \mathcal{B}, \mu)$ denote the product of measure spaces $(\Omega_1, \mathcal{B}_1, \mu_1)$ and $(\Omega_2, \mathcal{B}_2, \mu_2)$. For each the problems (a)-(c) below show that:

   (i) for μ_1-almost all x in Ω_1, the function
   $$f^x : \Omega_2 \to \mathbb{C} \text{ given by } f^x(y) = f(x, y)$$
   is $(\mathcal{B}_2, \mathcal{B}_{\mathbb{C}})$-measurable and in fact $f^x \in L^1(\Omega_2, \mathcal{B}_2, \mu_2)$;

   (i)$'$ for μ_2-almost all y in Ω_2, the function
   $$f_y : \Omega_1 \to \mathbb{C} \text{ given by } f^y(x) = f(x, y)$$
   is $(\mathcal{B}_1, \mathcal{B}_{\mathbb{C}})$-measurable and in fact $f_y \in L^1(\Omega_1, \mathcal{B}_1, \mu_1)$;

   (ii) the μ_1-almost everywhere defined function
   $$x \to \int f^x(y) d\mu_2(y)$$
   is $(\mathcal{B}_1, \mathcal{B}_{\mathbb{C}})$-measurable and in fact it is integrable with respect to μ_1;

   (ii)$'$ the μ_2-almost everywhere defined function
   $$y \to \int f_y(x) d\mu_1(x)$$
   is $(\mathcal{B}_2, \mathcal{B}_{\mathbb{C}})$-measurable and in fact it is integrable with respect to μ_2; and decide if :
   $$\int_\Omega f d\mu = \int_{\Omega_1} \left(\int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_2} \left(\int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y).$$

   (a) Consider $\Omega_1 = \Omega_2 = \mathbb{R}, \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_{\mathbb{R}}, \mu_1(dx) = dx, \mu_2(dy) = dy$ Let $A = \{(x, y) : |x - y| \leq 1\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
   $$f(x, y) = \begin{cases} 1 & \text{if } y \geq x, (x, y) \in A \\ -1 & \text{if } y < x, (x, y) \in A \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Consider $\Omega_1 = \Omega_2 = \mathbb{N} \cup \{0\}, \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{P}(\mathbb{N}), \mu_1(\cdot) = \mu_2(\cdot) =$ counting measure. Let $f : \mathbb{N}^2 \to \mathbb{R}$ be given by
   $$f(i, j) = \begin{cases} 1 & \text{for } j = i \\ -1 & \text{for } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (c) Consider $\Omega_1 = \Omega_2 = \mathbb{R}, \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_{\mathbb{R}}, \mu_1(dx) = dx, \mu_2(dy) = dy$
   $$f(x, y) = \begin{cases} \frac{\exp\{x^2y^2\} \sin(xy)}{1 + x^4y^4} & \text{for } (x, y) \in [-1,\ 1] \times [-1,\ 1] \\ 0 & \text{otherwise.} \end{cases}$$