C-12 Probability Theory **September 1, 2022** **Quiz 6** **Score**

Name ______________________ **Seat**

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1. Let $X \sim$ Binomial (n, p). Using the Chernoff bounds show that for $\delta > 0$

$$P(X \leq (1-\delta)np) \leq \exp(-\frac{\delta^2(np)}{2})$$

$\delta > 1$ then bound is obvious.

let $0 < \delta < 1$

let $\lambda < 0$

$$\mathbb{P}(X \leq (1-\delta)np) = \mathbb{P}(e^{\lambda X} \geq e^{\lambda(1-\delta)np})$$

(Markov Inequality) $$\leq e^{-\lambda(1-\delta)np} E[e^{\lambda X}]$$

$$= e^{-\lambda(1-\delta)np} (e^{\lambda} p + 1 - p)^n$$

set $\lambda = \ln(1-\delta) < 0 \Rightarrow$

$$\mathbb{P}(X \leq (1-\delta)np) = \left(\frac{1}{(1-\delta)^{(1-\delta)}}\right)^{np} (1-\delta p)^n$$

$$\leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{np}$$

Set: $f(\delta) = -\delta - (1-\delta)\ln(1-\delta) + \frac{\delta^2}{2}$

$$\left.\begin{aligned} f'(\delta) &= \ln(1-\delta) + \delta \\ f''(\delta) &= \frac{-1}{1-\delta} + 1 \end{aligned}\right\} \quad f''(\delta) < 0 \qquad 0 < \delta < 1$$

$f(0) = f'(0) = 0$

$\Rightarrow f(\delta) \leq 0$

$$\therefore \mathbb{P}(X \leq (1-\delta)np) \leq e^{-np\frac{\delta^2}{2}}$$