C-12 Probability Theory **September 1, 2022** **Quiz 4** **Score**

Name ______________________ **Seat**

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1. Provide an example of a random variables X, Y and $\alpha > 0$ such that

(a) $\mathbb{P}(X \geq \alpha) > \frac{E[X]}{\alpha}$.

(b) $\mathbb{E}[Y] < \infty$ and $\mathbb{E}[Y^{1.5}] = \infty$

Solution:

(a) $\Omega = [0,1)$, $\mathcal{B}_{[0,1)}$ - Borel σ-algebra, $\mathbb{P}(d\omega) = d\omega$

$X : \Omega \to [0,1)$

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in [0, \frac{1}{2}] \\ -1 & \text{if } \omega \in [\frac{1}{2}, 1) \end{cases}$$

- $E[X] = 0$, • $\mathbb{P}(X \geq \frac{1}{2}) = \frac{1}{2}$

$\Rightarrow \mathbb{P}(X \geq \frac{1}{2}) > 2E[X]$ □

(b) $X : \Omega \to [0,1)$

$$X(\omega) = \begin{cases} 0 & \omega = 0 \\ 2^{\frac{2}{3}k} & \omega \in (\frac{1}{2^k}, \frac{1}{2^{k-1}}] \quad \forall k \geq 1 \end{cases}$$

Let $X_m(\omega) = X \mathbb{1}_{(\frac{1}{2^m}, 1)}$ & $X_m \uparrow X$ as $m \to \infty$

$X_m \in S_+$, $E[X_m] = \sum_{k=1}^{m} 2^{\frac{2}{3}k} \frac{1}{2^k} = \sum_{k=1}^{m} \left(\frac{1}{2^{1/3}}\right)^k$

$X_m^{1.5} \in S_+$, $E[X_m^{1.5}] = \sum_{k=1}^{m} 2^k \frac{1}{2^k} = m$

clearly $E[X_m] \longrightarrow \sum_{k=1}^{\infty} \frac{1}{2^{k/3}} < \infty$ as $m \to \infty$

$E[X_m^{1.5}] \longrightarrow \infty$ as $n \to \infty$

$\therefore E[X] < \infty$ and $E[X^{1.5}] = \infty$