6. Distribution of Random Variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space and $X : \Omega \to \mathbb{R}$ be a random variable on it. The Distribution of X or the law of X is the Probability measure $\mathbb{Q}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ given by

$$\mathbb{Q}(A) = \mathbb{P}(X \in A), \quad \forall A \in \mathcal{B}_{\mathbb{R}}.$$

$$\| $$

$$\mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in A\})$$

$$\| $$

$$\mathbb{P}\cdot X^{-1}(A)$$

Distribution function of X is $F : \mathbb{R} \to [0,1]$ and is given

$$F(x) := \mathbb{P}(X \le x) \equiv \mathbb{Q}((-\infty, x])$$

$\boxed{\text{Ex:- HW 1 Problem 5}}$ $\Rightarrow$

ⓐ $F(x) \le F(y)$ if $x \le y$

ⓑ F is right-continuous

ⓒ F is not continuous at $x \in \mathbb{R} \iff \mathbb{Q}(\{x\}) > 0$

ⓓ $\lim_{x \to \infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$

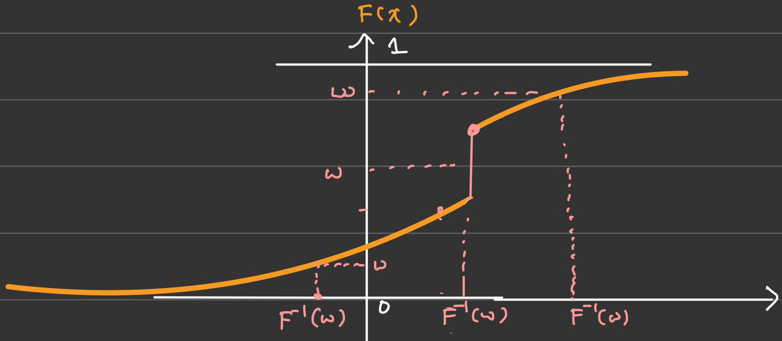
Conversely given $F: \mathbb{R} \to [0,1]$ satisfying (a) – (d).

Let $\Omega = [0,1]$, $\mathcal{B}_{[0,1]} \equiv$ Borel σ-algebra

$\mathbb{P}(d\omega) = d\omega$ (lebesgue measure)

Define: $X: \Omega \to \mathbb{R}$

$$X(\omega) = \inf\{x : \omega \leq F(x)\} \equiv F^{-1}(\omega)$$



By definition

$$\{X \leq x\} = \{\omega \in \Omega \mid X(\omega) \leq x\} \overset{Ex}{=} \{\omega \in \Omega \mid \omega \leq F(x)\}$$
$$= [0, F(x)]$$

$\Rightarrow$ X is a random variable

and $\mathbb{P}(X \leq x) =$ lebesgue measure $([0, F(x)])$

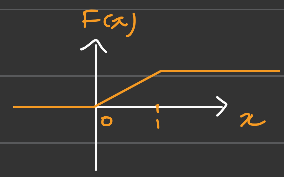
$= F(x)$ $\square$

Ex: Extend to other finite measures

Example 1: let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability Space.

- $X \sim$ Uniform $(0,1)$

$$F(x) = \begin{cases} 1 & x \geq 1 \\ x & 0 \leq x \leq 1 \\ 0 & x \leq 0 \end{cases}$$



$$\mathbb{P}(X \leq x) = F(x)$$

Ex: F is piecewise differentiable (F is absolutely continuous)

if $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

then $\mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$

If $X : \Omega \to \mathbb{R}$ is a random variable with distribution function $F : \mathbb{R} \to [0,1]$ being absolutely continuous then there exist $f : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(y)\, dy \quad \&$$

$f \equiv$ Probability density function of X.

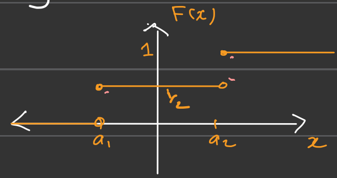
$X \equiv$ (absolutely) Continuous r.v.

Example 2:- $X \sim \text{Uniform}(a_1, a_2)$ $\quad a_1 < a_2$

i.e. $\mathbb{P}(X = a_1) = \frac{1}{2}$, $\mathbb{P}(X = a_2) = \frac{1}{2}$

$F(x) = \mathbb{P}(X \leq x)$ is given by

$$= \begin{cases} 0 & x < a_1 \\ 1/2 & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \end{cases}$$



X is Discrete random variable if

Range(X) is Countable

(i) $\mathbb{P}(X = x_i) > 0 \quad \forall\, x_i \in \text{Range}(X)$

(ii) $\displaystyle\sum_{i \in \text{Range}(X)} \mathbb{P}(X = x_i) = 1$

Example 3:

$$X \equiv \begin{cases} \text{lifetime of a bulb} \\ \text{waiting time in a Queue} \end{cases}$$

X has the property

"Probability waiting time exceeds y"

$\equiv$

"Probability waiting time exceeds x+y given that it has exceeded x"

<u>Model</u> : $\quad 1 - F(y) = \dfrac{1 - F(x+y)}{1 - F(x)} \quad \forall\, x, y \in [0, \infty)$

$$\Leftrightarrow \quad 1 - F(x) = e^{-\lambda x} \quad \forall\, x \in [0, \infty) \text{ for some } \lambda > 0$$

$$\Leftrightarrow \quad F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$

X - random variable has Exponential (λ) distribution, (Ex), with probability density function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\lambda x} & x > 0 \end{cases}$$

<u>Example 4</u> :

$X \sim$ Normal (μ, σ^2) if X has p.d.f. given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad x \in \mathbb{R}.$$

7 Expectation of a Random Variable

-(integration for random variables)

(Review from Measure Theory)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space

$X: \Omega \to \mathbb{R}$ be a random variable.

Definition 1: A random variable X is Simple if

$\text{Range}(X) < \infty$.

Remark 1:-

(Ex) (i) X is simple $\iff$

$\exists\, n \geq 1$, $a_i \in \mathbb{R}$, $E_i \in \mathcal{B}_{\mathbb{R}}$ such that $E_i \cap E_j = \phi$

$1 \leq i, j \leq n$ and

$$X = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i} \quad - \quad (1)$$

$a_i \neq a_j$ & $E_i = X^{-1}(\{a_i\}) \equiv$ Canonical

(Ex) (ii) Decomposition in (1) is not Unique

but if $\{E_i\}_{i=1}^{n} \in \mathcal{F}$ $\quad E_i \cap E_j = \phi \quad i \neq j$

$\{F_i\}_{i=1}^{n} \in \mathcal{F}$ $\quad F_i \cap F_j = \phi \quad i \neq j$

$$\Omega = \bigsqcup_{i=1}^{n} E_i = \bigsqcup_{i=1}^{n} F_i$$

then $\sum_{i=1}^{n} \mathbb{P}(E_i) = \sum_{i=1}^{n} \mathbb{P}(F_i)$

Definition 2 :- Let

$$S_+ := \{ X : \Omega \to [0,\infty) \mid X \text{ is simple r.v.} \}$$

$X \in S_+$ and $X = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}$ (Remark 7(i))

for some $a_i \in \mathbb{R}$ and $E_i \in \mathcal{F}$ $E_i \cap E_j = \phi$

Expectation of X as

$$E[X] \equiv \int X \, d\mathbb{P} := \sum_{i=1}^{n} a_i \, \mathbb{P}(E_i) \in [0,\infty)$$

Remark 2 :-

(i) [Remark 7.1 (ii)] $\int X \, d\mathbb{P}$ is well defined for $X \in S_+$

(ii) $X \in S_+$, $Y \in S_+$ $\alpha, \beta \geq 0$

$\alpha X + \beta Y \in S_+$ and

$$\int (\alpha X + \beta Y) \, d\mathbb{P} = \alpha \int X \, d\mathbb{P} + \beta \int Y \, d\mathbb{P} \quad -(2)$$

(iii) $S_+ \ni X \longrightarrow \int X \, dP$ is the unique map such that

- $\int 1_E \, dP = P(E) \quad \forall E \in \mathcal{F}$

and (2) holds.

Suppose $\boxed{X \leq Y}$ (Pointwise) & $X, Y \in S_+$.

Then $h = Y - X \geq 0$ (Pointwise) $\in S_+$

$\Rightarrow \int h \, dP \geq 0$

$\Rightarrow \int Y \, dP - \int X \, dP \geq 0$

$\boxed{\Rightarrow} \quad \boxed{\int X \, dP \leq \int Y \, dP}$

Proposition 1 :- Let $X : \Omega \to [0, \infty)$ be a random variable. Then $\exists \{X_n\}_{n \geq 1}$

- $X_n \in S_+$,
- $X_n \leq X_{n+1}$
- $X_n \longrightarrow X$ on Ω

$\Bigg\}$ (pointwise)

Proof: Proposition 2.1.4 in [AS] $\square$

$\mathcal{L}^0_+ = \{ X : \Omega \to [0,\infty) \mid X \text{ is a r.v.} \}$

Definition 3: For $X \in \mathcal{L}^0_+$, define expectation of X:

$$E[X] = \int X \, d\mathbb{P} := \sup \left\{ \int s \, d\mathbb{P} \;\middle|\; 0 \le s \le X, \; s \in S_+ \right\}$$

Remark 3:-

- $\mathcal{L}^0_+ \ni X \longrightarrow \int X \, d\mathbb{P}$ is the unique map that satisfies

  1. $\int \mathbb{1}_E \, d\mathbb{P} = \mathbb{P}(E) \quad \forall E \in \mathcal{F}$
  2. $\alpha, \beta \in (0,\infty) \qquad X, Y \in \mathcal{L}^0_+$
     $$\int (\alpha X + \beta Y) \, d\mathbb{P} = \alpha \int X \, d\mathbb{P} + \beta \int Y \, d\mathbb{P}$$
  3. If $s_n \in S_+$, $s_n \uparrow X$ pointwise then
     $$\int X \, d\mathbb{P} = \sup_n \int s_n \, d\mathbb{P}.$$

Definition 4: $X : \Omega \to \mathbb{R}$ is a random variable.

Let $X_+ = \max(X, 0) \qquad X_- = -\min(X, 0)$

If $\int X_+ \, d\mathbb{P} < \infty$ or $\int X_- \, d\mathbb{P} < \infty$ then we define $\int X \, d\mathbb{P} = \int X_+ \, d\mathbb{P} - \int X_- \, d\mathbb{P}$.

$$E[X] = E[X_+] - E[X_-]$$

otherwise we say $E[X]$ does not exist.

Recall : August 25^{th}, 2022

- $X \in S_+$ and $X = \sum_{i=1}^{n} a_i 1_{E_i}$ (for some $a_i \in \mathbb{R}$ and $E_i \in \mathcal{F}$ $E_i \cap E_j = \phi$

(Mean)
Expectation of X as

$$E[X] \equiv \int X \, d\mathbb{P} := \sum_{i=1}^{n} a_i \, \mathbb{P}(E_i) \in [0, \infty)$$

- $E[X] \equiv \int X \, d\mathbb{P} := \sup \left\{ \int s \, d\mathbb{P} \,\middle|\, 0 \leq s \leq X \,, s \in S_+ \right\}$

- $\mathcal{L}_+^0 \ni X$ ——— $\int X d\mathbb{P}$ is the unique map that satisfies

① $\int 1_E \, d\mathbb{P} = \mathbb{P}(E) \quad \forall \; E \in \mathcal{F}$

② $\alpha, \beta \in (0, \infty) \qquad X, Y \in \mathcal{L}_+^0$

$$\int (\alpha X + \beta Y) \, d\mathbb{P} = \alpha \int X d\mathbb{P} + \beta \int Y d\mathbb{P}$$

③ If $s_n \in S_+$ $s_n \uparrow X$ pointwise then

$$\int X d\mathbb{P} = \sup_n \int s_n \, d\mathbb{P}.$$

- $E[X] = E[X_+] - E[X_-]$

August 25^{th} 2022 (Contd. of Expectation ...)

Example 1

- $X \sim \text{Uniform}(a,b)$ X has p.d.f given by

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X \in A) = \int_A f(x)\,dx \qquad \forall A \in \mathcal{B}_{\mathbb{R}}$$

$$E[X] \overset{(Ex)}{=} \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{b+a}{2}$$

- $X \sim \text{Uniform}(\{a_1, a_2, \dots, a_n\})$

$$\mathbb{P}(X = a_i) = p_i > 0 \qquad 1 \le i \le n$$

$$\sum_{i=1}^{n} p_i = 1$$

$$E[X] = \sum_{i=1}^{n} a_i p_i \equiv \text{weighted average of } \{a_1, \dots, a_n\}$$

Definition 5 :: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. $X: \Omega \to \mathbb{R}$ be a random variable such that $\mu = \mathbb{E}[X] < \infty$. Then the variance of X is defined as

$$\sigma^2 \equiv Var[X] := E[(X-\mu)^2]$$
$$= E[X^2] - \mu^2$$

standard deviation of X is defined as

$$\sigma \equiv SD[X] = \sqrt{Var[X]}$$

Remark 4:

$\mu \equiv E[X]$ = "centre" of the Range (X)

$\sigma \equiv SD[X]$ = "spread" of the Range (X) around the mean.

(larger σ "$\Longleftrightarrow$" X takes values far away from μ with high probability)

μ, σ – have the same units

$(\mu - 3\sigma, \mu + 3\sigma)$ = "Effective" range of X.

8 Key Inequalities

This section shall contain some proofs of very fundamental inequalities.

Proposition 1 (Markov's Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. If X is a non-negative random variable, then for all $\alpha > 0$

$$\mathbb{P}(X > \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

Proof:-

Let $Z : \Omega \to \mathbb{R}$

$$Z(\omega) = \begin{cases} \alpha & X(\omega) \geq \alpha \\ 0 & X(\omega) < \alpha \end{cases}$$

- $Z \in S_+$ (why?)
- $Z \leq X$

(Ex.)

$\Rightarrow$ $\mathbb{E}[Z] \leq \mathbb{E}[X]$

$\therefore$ $\alpha\, \mathbb{P}(X \geq \alpha) \leq \mathbb{E}[X]$

$$\Rightarrow \quad \mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha} \qquad \square$$

Remark 1: Its okay to have $\mathbb{E}[X] = \infty$.

For general random variables

Proposition 5.1.2 (Chebyschev Inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. If X is a random variable, with finite mean μ.

Then for all $\alpha \geq 0$

$$\mathbb{P}(|X - \mu| > \alpha) \leq \frac{Var(X)}{\alpha^2}$$

Proof:- Let $Z = |X - \mu|^2$

Then $Z \geq 0$ and using Proposition 1 we have

$$\mathbb{P}(Z \geq \alpha^2) \leq \frac{E[Z]}{\alpha^2}$$

But $\{|X-\mu| \geq \alpha\} = \{Z \geq \alpha^2\}$

$$\Rightarrow \mathbb{P}(|X-\mu| \geq \alpha) \leq \frac{E(X-\mu)^2}{\alpha^2} = \frac{Var(X)}{\alpha^2}$$

$\square$

Remark 2: Take $\alpha = k\sigma$, for $k \geq 1$ in chebyschev inequality then $\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$.

i.e. $\mathbb{P}(X \notin (\mu - k\sigma, \mu + k\sigma)) \leq \frac{1}{k^2}$

$\forall\, k \geq 1$.

Proposition 3 (Cauchy-Bunyakovsky-Schwarz inequality)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. Let X, Y be random variables such that $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$. Then

$$\mathbb{E}|XY| \leq \sqrt{\mathbb{E}[X^2]\, \mathbb{E}[Y^2]}$$

Proof:- $Z = \frac{|X|}{\sqrt{E[X^2]}}$, $W = \frac{|Y|}{\sqrt{E[Y^2]}}$

Both Z, W are measurable functions of X, Y $\Rightarrow$ Z, W are random variables.

$$(Z-W)^2 \geq 0$$

$$\Leftrightarrow \quad Z^2 + W^2 \geq 2ZW$$

$$\Rightarrow \mathbb{E}[ZW] \leq \frac{1}{2} \mathbb{E}[Z^2 + W^2]$$

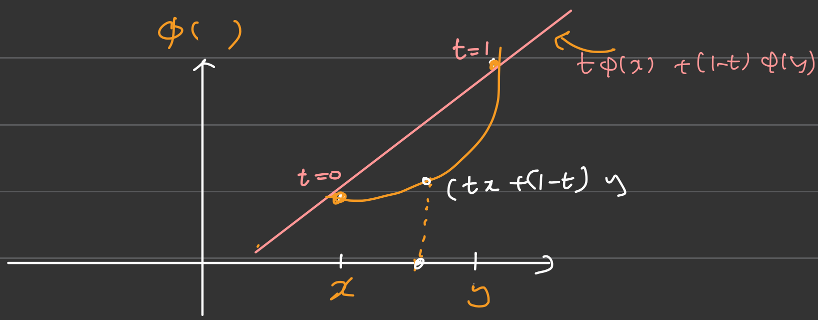
$$\Rightarrow \mathbb{E}|XY| \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]} \qquad \square$$

Proposition 4 (Jensen's Inequality):
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space and X be a random variable with finite mean $\mathbb{E}[X]$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function such that $\mathbb{E}[\phi(X)]$ exists.

Then $\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X])$

Proof:- Now ϕ is convex $\Leftrightarrow$

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$
$$\forall t \in (0,1) \quad \forall x, y \in \mathbb{R}.$$

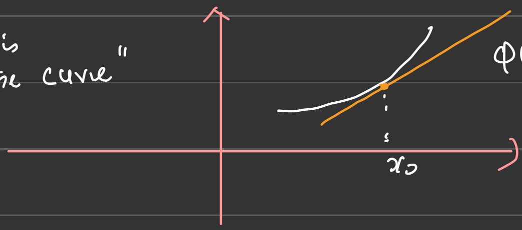


lemma 1: Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. Then $\forall x_0 \in \mathbb{R}$

$\exists\ c \equiv \partial\phi(x_0)$ s.t.

$$\phi(x) \geq \phi(x_0) + c(x - x_0)$$

"Tangent is below the curve"



Proof of lemma 1: let $x < y < z$

$$\Rightarrow \quad y = tx + (1-t)z \qquad \text{for } t \in [0,1]$$

$$\Rightarrow \quad t = \frac{y - z}{x - z} \in (0,1)$$

ϕ - convex $\Rightarrow$

$$\phi(tx + (1-t)z) \leq t\phi(x) + (1-t)\phi(z)$$

$$\Leftrightarrow \quad \frac{\phi(x) - \phi(y)}{x - y} \leq \frac{\phi(z) - \phi(y)}{z - y} \quad -(1)$$

$$\forall\ x < y < z$$

$\therefore\ k \geq 1 \qquad a_k = \dfrac{\phi(x_0 - \frac{1}{k}) - \phi(x_0)}{-1/k}$

$m \geq 1$ $\quad b_m = \dfrac{\phi(x_0+\frac{1}{m}) - \phi(x_0)}{1/m}$

By (1) : $a_k \leq a_{k+1} \quad \forall\ k \geq 1$

$b_m \geq b_{m+1} \quad \forall\ m \geq 1$

$a_k \leq b_m \quad \forall\ k \geq 1,\ m \geq 1$

$\Rightarrow \quad \alpha := \lim_{k\to\infty} a_k \quad \beta := \lim_{m\to\infty} b_m$

exists and

$\alpha \leq \beta$

: If $\alpha = \beta$ then $c = \alpha = \beta$ (if f is differentiable at x_0)

otherwise choose any $c \in (\alpha, \beta)$

Also, $x < x_0$ by (1)

$$\frac{\phi(x) - \phi(x_0)}{x - x_0} \leq \alpha \leq c$$

Similarly, $x > x_0$ by (1)

$$\frac{\phi(x) - \phi(x_0)}{x - x_0} \geq \beta \geq c$$

$\Rightarrow \forall\ x \in \mathbb{R} \quad \phi(x) \geq c(x - x_0) + \phi(x_0)$ $\square$

Proof:- Let $x_0 = E[X]$ By lemma 1

$\exists\, c \in \mathbb{R}$ s.t.

$$\phi(X) \geq \phi(x_0) + c(X - X_0)$$

$$\Rightarrow \mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X]) + 0$$

$\square$