Recall:-

[Borel – Cantelli lemma]

$(\Omega, \mathcal{F}, \mathbb{P})$ · Probability space

$\{A_n\}_{n\geq 1} \in \mathcal{F}$

$\bigcap_{N\geq 1} \bigcup_{n\geq N} A_n$

① $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(\limsup_{n\to\infty} A_n) = 0$

② $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, $A_n$ independent, $\mathbb{P}(\limsup_{n\to\infty} A_n) = 1$

Remarks:

- $H_n = \{$ nth toss is a head in infinite tosses of a coin $\}$

[BC] $\Rightarrow$ $\mathbb{P}(H_n \text{ i.o.}) = 1$

- $B_n = \underbrace{H_{2^n+1} \cap H_{2^n+2} \cdots \cap H_{2^n+a_n}}_{\text{"run of Heads"}}$

Q: $\mathbb{P}(B_n \text{ i.o.}) = 1, 0$ ?

Ex: · $a_n = \log_2(n)$ $\Rightarrow$ $B_n$ are independent

$\mathbb{P}(B_n) \sim \frac{1}{n}$ $\overset{[B.C.]}{\Rightarrow}$ $\mathbb{P}(B_n \text{ i.o.}) = 1$

· $a_n = 2\log_2(n)$

$\mathbb{P}(B_n) \sim \frac{1}{n^2}$ $\overset{[B.C.]}{\Rightarrow}$ $\mathbb{P}(B_n \text{ i.o.}) = 0$

18th – August

## 5. Kolmogorov 0-1 law

$(\Omega, \mathcal{F}, \mathbb{P})$ – Probability space

$A_n \in \mathcal{F} \quad \forall\, n \geq 1$

$\sigma(\{A_m : m \geq n\}) :=$ Smallest $\sigma$-Algebra Containing $\{A_m : m \geq n\}$

[ Ex:- Existence from Measure Theory ]

**Definition 1:-** The Tail $\sigma$-field corresponding to $\{A_n\}_{n\geq 1}$, $A_n \in \mathcal{F}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as

$$\mathcal{T} := \bigcap_{n=1}^{\infty} \sigma(\{A_m \mid m \geq n\})$$

**Remarks:-**

- (Ex) $\mathcal{T}$ – $\sigma$-field $\subseteq \mathcal{F}$.
- $A \in \mathcal{T} \implies A \in \sigma(A_n, A_{n+1}, \dots) \quad \forall\, n \geq 1$

  in particular $A_n$ will not worry about finite number of events $A_n$.

**Examples of $A \in \mathcal{T}$ :-**

- $H_n = \{$ nth toss is a head in infinite tosses of a coin $\}$

  Ex: $\limsup_{n\to\infty} H_n \in \mathcal{T}$

$\limsup\limits_{n\to\infty} A_n \in \tau$

- $X_n = \begin{cases} 1 & \text{if } A_n \text{ occurs} \\ 0 & \text{if } A_n \text{ does not occur} \end{cases}$

$A = \{ \sum_{n=1}^{\infty} X_n \text{ converges} \} \in \tau$

**Theorem 1: [Kolmogorov – Zero–One law]** On $(\Omega, \mathcal{F}, \mathbb{P})$ – a probability space if events $A_n \in \mathcal{F}$ $\forall n \geq 1$ are independent then

$$A \in \tau \left( := \bigcap_{n=1}^{\infty} \sigma(\{A_m \mid m \geq n\}) \right)$$

has $\mathbb{P}(A) \in \{0, 1\}$.

**Proof:-** let $A \in \tau$

$\Rightarrow \quad A \in \sigma(A_n, A_{n+1}, \dots) \quad \forall n \geq 1$

- **Fact 1:** As $\{A_n\}_{n \geq 1}$ is an independent sequence of events

$\forall n \geq 1 \quad A, A_1, \dots, A_{n-1}$ are independent

Ex:- Use Fact 2

$\Rightarrow \quad \{A, A_n : n \geq 1\}$ is an independent collection of events — (1)

**Fact 2:** Assume (1). Then

$S \in \sigma(\{A_n : n \geq 1\}) \Rightarrow A$ and $S$ are independent

However $A \in \mathcal{T} \Rightarrow A \in \sigma(\{A_n : n \geq 1\})$

$\Rightarrow A$ is independent of $A$ !

$$\Rightarrow \quad P(A \cap A) = \mathbb{P}(A)\,\mathbb{P}(A)$$

$$\Rightarrow \quad \mathbb{P}(A) = (\mathbb{P}(A))^2$$

$$\Rightarrow \quad \mathbb{P}(A) \in \{0,1\}.$$

$\square$

**Proof of Fact 2:-**

$\{A, A_1, A_2, \dots\}$ is a collection of mutually independent events.

Let $S \in \sigma(\{A_n : n \geq 1\})$

To show: $\mathbb{P}(A \cap S) = \mathbb{P}(A)\,\mathbb{P}(S)$ —(2)

- $\mathbb{P}(A) = 0 \Rightarrow$ (2) holds trivially.

- $\mathbb{P}(A) > 0$

$$\mathcal{A} = \left\{ \bigcap_{j=1}^{n} D_j \;\middle|\; D_j = A_{i_j}^{\varepsilon_j},\ \varepsilon_j \in \{0,1\},\ A_{i_j}^1 = A_{i_j} \ \& \ A_{i_j}^0 = A_{i_j}^c,\ n \geq 1 \right\}$$

Ex: $\mathcal{A}$ is a field / algebra

observe: $S \in \mathcal{A}$

- By independence (Definition)

$$\mathbb{P}(S \cap A) = \mathbb{P}(S)\,\mathbb{P}(A)$$

$$\Rightarrow \quad \mathbb{P}(S) = \frac{\mathbb{P}(S \cap A)}{\mathbb{P}(A)} \quad -(3)$$

- Construct $\mathbb{Q}$ on $(\Omega, \sigma(A_n : n \geq 1), \mathcal{F})$

$S \in \sigma(\{A_n : n \geq 1\})$

$$\mathbb{Q}(S) := \frac{\mathbb{P}(S \cap A)}{\mathbb{P}(A)} \quad -(4)$$

Ex: $\mathbb{Q}$ is indeed a Probability.

By (3) and (4) we have

$$\mathbb{Q}(S) = \mathbb{P}(S) \quad \forall\, S \in \mathcal{A}.$$

Ex: Extension Theorem $\Rightarrow$

$$\Rightarrow \mathbb{Q}(S) = \mathbb{P}(S) \;\forall\, S \in \sigma(A_n : n \geq 1) :\equiv \sigma(\mathcal{A})$$

$$\Rightarrow \frac{\mathbb{P}(S \cap A)}{\mathbb{P}(A)} = \mathbb{P}(S) \quad \forall\, S \in \sigma(\{A_n : n \geq 1\})$$

$\Rightarrow \quad \mathbb{P}(S \cap A) = \mathbb{P}(A)\,\mathbb{P}(S) \quad \forall\, S \in \sigma(\{A_n : n \geq 1\})$

$\Rightarrow$ $S$ and $A$ are independent $\quad \blacksquare$

$\forall\, S \in \sigma(\{A_n : n \geq 1\})$

**Bond - Percolation on $\mathbb{Z}^d$**, $d \geq 2$

$$E_d := \{ \{x,y\} \mid x,y \in \mathbb{Z}^d \ \& \ |x-y| = 1 \}$$

- nearest-neighbour edges in $\mathbb{Z}^d$.

Fix $0 \leq p \leq 1$; declare each edge $e \in E_d$ to be

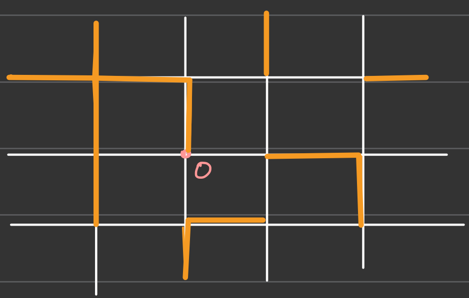
- open with probability $p$
- closed with probability $1-p$

in an i.i.d. manner

[ each $e \in E_d$ is independently open or closed with probability $p$ ]

References

- Grimett
- Bollobas & Riordan
- Broadbent Hammersley



**Question:-** $E = \left\{ \begin{array}{l} \text{there exists an infinite connected component of open} \\ \text{edges} \end{array} \right\}$

$\Omega = \{0,1\}^{E_d}$ ; $0$ "closed", $1$ "open"

$\mathcal{A} :=$ σ-field generated cylinder sets

$$\mathbb{P}_p = \bigotimes_{e \in E_d} \mu_e$$

$$\mu_e(\omega(e)=1) = p \; ; \; \mu_e(\omega(e)=0) = 1-p$$

$$\mathbb{P}_p(E) = ?$$

**Answer:-**

$$p = 0 \implies \mathbb{P}_p(E) = 0 \qquad -(1)$$

$$p = 1 \implies \mathbb{P}_p(E) = 1 \qquad -(2)$$

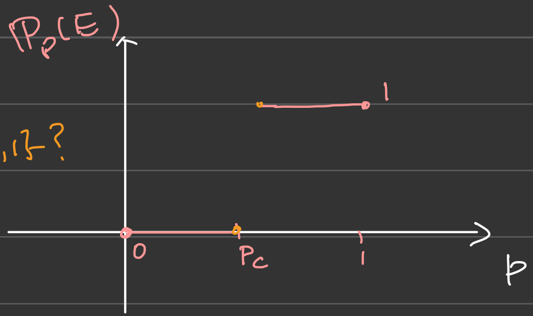
**Ex:-**

- $E \in \mathcal{A}$.
- $A_e = \{e = 1\}$ $\quad e \in E_d$
  $\{A_e\}_{e \in E_d}$ are independent
- $E \in \tau :=$ tail σ-field corresponding to $A_e$

$$\xRightarrow[\text{0-1 law}]{\text{Kolmogorov}} \quad \mathbb{P}_p(E) \in \{0,1\}. \qquad -(3)$$

Ex: (Intuitively obvious) $0 \le p_1 \le p_2 \le 1$

$$\Rightarrow \quad \mathbb{P}_{p_1}(E) \le \mathbb{P}_{p_2}(E) \qquad -(4)$$




(1) − (4)

I

Open: $\mathbb{P}_{p_c}(E) \in \{0,1\}$?

$p_c :=$ critical probability of percolation

[Kesten] $d=2$ $\quad p_c = \frac{1}{2}$. $\quad \mathbb{P}_{p_c}(E) = 0$.

[Hara - Slade] $d \ge 11$ $\quad p_c = \quad \mathbb{P}_{p_c}(E) = 0$

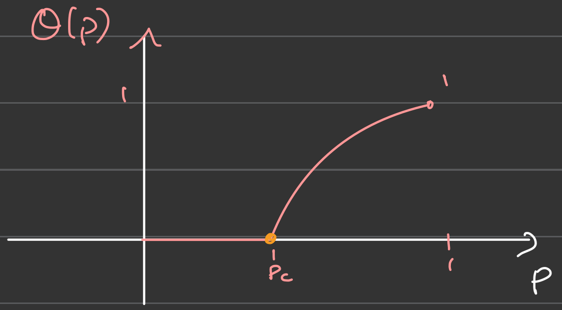
$d \to \infty \quad \sim \frac{1}{2d} + \left(\frac{1}{2d}\right)^2 + \cdots$

$3 \le d \le 10$ − open problem

$$C_0^\infty = \{ 0 \in \mathbb{Z}^d \in E \} = \{ 0 \rightsquigarrow \infty \text{ along open edges in } \mathbb{Z}^d \}$$

Ex:- $\mathbb{P}_p(E) = 1 \iff \underbrace{\mathbb{P}_p(C_0^\infty)}_{\theta(p)} > 0$

Compare with graph $\textcircled{I}$




**Question**

"Behaviour near $p_c$"

$$|\Theta(p) - \Theta(p_c)| \approx |p - p_c|^{\gamma} \qquad \gamma = ?$$