Recall :-

Measure Theory :-

Ω := any non empty set

[Referred to as :- Sample Space ≡ Set of all possible outcomes of an Experiment.]

$\mathcal{F} \subseteq \mathcal{P}(\Omega)$ (Power set of Ω)

(i) $\phi, \Omega \in \mathcal{F}$

[σ-algebra
σ-field]

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_n \in \mathcal{F} \;\forall\; n \geq 1$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

[Referred to as :- Event (subcollection of outcomes)]

$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$

① $P(\Omega) = 1$

② $E_n \in \mathcal{F}$, $E_n \cap E_\ell = \phi$ for $n \neq \ell$

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$$

[Referred to as :- Probability = function on sets satisfying Axioms ① and ②]

Discussed: Need for each abstraction and examples of

$(\Omega, \mathcal{F}, \mathbb{P})$ - Probability space.

Bernoulli(p)	Uniform$(0,1)$
Binomial(n,p)	Normal$(0,1)$
Poisson(λ)	Mixtures

· Assume $([a,b], \mathcal{B}_{[a,b]}, dx)$

↳ lebesgue measure Exists.

2 Random Variables :-

Definition 1 :- let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. $X : \Omega \to \mathbb{R}$ is a random variable if

$$\{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F} \quad \forall\, x \in \mathbb{R} \quad (1)$$

Remark 1 :-

ⓐ (Recall) $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable if $\{x \in \mathbb{R} \mid f(x) \leq a\} \in \mathcal{B}_{\mathbb{R}} \quad \forall\, a \in \mathbb{R}$

ⓑ (Notation) $\{X \leq x\} \equiv \{\omega \in \Omega \mid X(\omega) \leq x\}$

ⓒ (Exercise) X is a random variable if $X^{-1}(B) \in \mathcal{F} \quad \forall\, B \in \mathcal{B}_{\mathbb{R}}$.

- will ensure that we can talk about :- $\mathbb{P}(X \in B) \quad \forall\, B \in \mathcal{B}_{\mathbb{R}}$.

ⓓ (Non-measurable) Not all functions $X : \Omega \to \mathbb{R}$ are measurable.
(Problem 1 : Worksheet 1)

Example 1 : $X \sim \text{Uniform}(S)$ $|S| < \infty$ $S \subseteq \mathbb{R}$

$\Omega = S$

$\mathcal{F} = \mathcal{P}(\Omega)$

$\mathbb{P} : \mathcal{F} \to [0,1]$ by $\mathbb{P}(A) = \frac{|A|}{|S|}$

$(\Omega, \mathcal{F}, \mathbb{P})$ Probability Space

$X : \Omega \to \mathbb{R}$ $X(\omega) = \omega$

$\mathbb{P}(X = i) = \frac{1}{|S|}$ $\forall\, i \in S$

X Random variable

X - is a point uniformly chosen in S.

- random variable, as any function is.

- $S = \{1, 2, 3, 4, 5, 6\}$

  X = roll of outcome of a dice.

- $S = \{0, 1\}$

  $Z = 5X + 3$ Range $(Z) = \{3, 8\}$

  $\mathbb{P}(Z = 3) = \frac{1}{2} = \mathbb{P}(Z = 8)$

- $S = \{-1, 0, 1\}$

  $Y = X^2$ Range $(Y) = \{0, 1\}$

  "Distribution" of Y

  $\mathbb{P}(Y = 0) = \frac{1}{3}$

  $\mathbb{P}(Y = 1) = \frac{2}{3}$

Example 2: $X \sim \text{Uniform}(0,1)$

$\Omega = [0,1]$ $\quad \mathcal{F} = \mathcal{B}_{[0,1]}$ $\quad \mathbb{P}(d\omega) = d\omega$ $\cdot$ lebesgue on $(0,1)$

$X : \Omega \to \Omega$ $\quad X(\omega) = \omega$ $\quad$ (clearly Random variable)

$$\mathbb{P}(a < X < b) = b - a \qquad 0 \le a \le b \le 1$$

(Distribution of X is uniform on $(0,1)$
X - a point chosen uniformly at random in $(0,1)$)

- $Y = 4X + 3$

let $a \in \mathbb{R}$.

$$\{\omega \in \Omega \mid Y(\omega) \le a\} = \{\omega \in \Omega \mid X(\omega) \in \tfrac{a-3}{4}\} = \begin{cases} \phi & a < 3 \\ [0, \frac{a-3}{4}] & 3 \le a \le 7 \\ [0,1] & a \ge 7 \end{cases} \quad \in \mathcal{F} \text{ in all cases}$$

($Y : \Omega \to \mathbb{R}$ is a random variable)

Range$(Y) = (3,7)$

$$\mathbb{P}(Y \le y) = \mathbb{P}\left(X \le \frac{y-3}{4}\right) = \begin{cases} 0 & \text{if } y \le 3 \\ 1 & \text{if } y \ge 7 \\ \frac{y-3}{4} & \text{if } 3 < y < 7 \end{cases}$$

Ex: "Y chooses a point uniformly in $[3,7]$".

If $\mathbb{Q} = \mathbb{P} \cdot Y^{-1}$ on $([3,7], \mathcal{B}_{[3,7]})$

"Distribution of Y"

$$\mathbb{Q}([c,d]) = \frac{d-c}{4} \; ; \quad 3 \le c \le d \le 7$$

normalised lebesgue measure

$Y \sim \text{Uniform}(3,7)$

Definition 2 :-

- $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is Borel-measurable if $\{x \in \mathbb{R}^n \mid f(x) \le a\} \in \mathcal{B}_{\mathbb{R}^n}$ - Borel σ-algebra in $\mathbb{R}^n$.

Facts from Measure Theory

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space

$n \geq 1$ $\quad X_1, X_2, \ldots, X_n$ be random variables.

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Borel measurable function.

$Y : \Omega \longrightarrow \mathbb{R}$ given by

$$Y = f(X_1, X_2, \ldots, X_n)$$

is also a random variable.

Remark :- Choosing $n \geq 1$ and $f : \mathbb{R}^n \to \mathbb{R}$

are all random variables
- $cX_1 + d$ for $c, d \in \mathbb{R}$
- $X_1 \cdot X_2$
- X_1^2
- $X_1 + X_2$

- limits of random variables are random variables

let $X_1, X_2, \dots, X_n$ be a sequence of random variables such that

$X_n \longrightarrow X \quad \forall\, \omega \in \Omega$ as $n \to \infty$. — (2)

let $m \geq 1$ $\omega \in \Omega$

$\exists\, N$ st $|X_n(\omega) - X(\omega)| \leq \frac{1}{m} \quad \forall\, n \geq N$

let $a \in \mathbb{R}$

$$\{\omega \in \Omega : X(\omega) \leq a\}$$

$$(Ex) = \{\omega \in \Omega : \forall\, m \geq 1 \;\; \exists\, N \;\; X_n(\omega) \leq a + \frac{1}{m}, \; \forall\, n \geq N\}$$

$$= \bigcap_{m \geq 1} \bigcup_{N \geq 1} \bigcap_{n \geq N} \{\omega \in \Omega \mid X_n(\omega) \leq a + \frac{1}{m}\}$$

$\in \mathcal{F}$

As $a \in \mathbb{R}$ arbitrary $\Rightarrow$ X is r.v.

3 **Independence :-**

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space and $A, B \in \mathcal{F}$ be two events.

Question: Does the occurence (non-occurence) of B affect the Probability of A ?

Example 1 :- Suppose we toss a coin three times

$$\Omega = \{h, t\}^3, \quad \mathcal{F} = \mathcal{P}(\Omega)$$

$\mathbb{P} : \mathcal{F} \to [0,1]$ by

$$\mathbb{P}(A) = |A|/8. \qquad A \in \mathcal{F}$$

$A = \{\text{first toss is a head}\} = \{hhh, hht, hth, htt\}$

$B = \{\text{second toss is a head}\} = \{hhh, hht, tht, thh\}$

$\mathbb{P}(A) = \frac{1}{2}$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{3}{4} = \frac{1}{2}$$

$$\mathbb{P}(A \mid B^c) = \frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(B^c)} = \frac{3}{4} = \frac{1}{2}$$

$$\left.\begin{aligned} &\mathbb{P}(A) = \mathbb{P}(A \mid B^c) = \mathbb{P}(A \mid B) \\ \text{Ex:}\quad &\mathbb{P}(B) = \mathbb{P}(B \mid A) = \mathbb{P}(B \mid A^c) \end{aligned}\right\} - (1)$$

Definition 1 : Two events A and B are independent iff

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \qquad -(2)$$

- $\forall n \geq 1$ its tempting to define independence of $A_1, A_2, \dots, A_n$ as

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \prod_{i=1}^{n} \mathbb{P}(A_i) \qquad -(3)$$

Example 2 :- Toss a fair coin two times.

$A_1 = \{hh, ht\}$, $A_2 = \{hh, ht\}$, $A_3 = \{hh, th\}$

$$\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \frac{1}{2}$$

$$\mathbb{P}(A_i \cap A_j) = \frac{1}{4} \qquad i \neq j$$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = 1/4$$

$\therefore$ pairwise independence holds as per Definition 1. However (3) does not hold for $n = 3$.

Definition 2: A finite collection of events $A_1, A_2, \ldots, A_n$ is mutually independent if

(4) — $$\mathbb{P}(E_1 \cap E_2 \cap E_3 \cdots \cap E_n) = \prod_{i=1}^{n} \mathbb{P}(E_i)$$

where $E_i = A_i$ or A_i^c $i = 1, 2, \ldots, n$.

An arbitrary collection of events A_t where $t \in I$ for some index set I is mutually independent if every finite subcollection is mutually independent.

Remark 1:-

- Mutual independence – (4) – contains 2^n equations.

- (4) includes (3) and will ensure pairwise independence $\Leftarrow$ Mutual Independence.

- It is easy to check that (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (1) for $n=2$

Recall:

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. $X: \Omega \to \mathbb{R}$ is a random variable if

$$\{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F} \quad \forall\ x \in \mathbb{R} \quad (1)$$

- $f: \mathbb{R} \to \mathbb{R}$ Borel m'ble $\Rightarrow$ $f(X)$ is R.v.

A finite collection of events $A_1, A_2, \dots, A_n$ is mutually independent if

$$(4) \quad - \quad \mathbb{P}(E_1 \cap E_2 \cap E_3 \cdots \cap E_n) = \prod_{i=1}^{n} \mathbb{P}(E_i)$$

where $E_i = A_i$ or A_i^c $\quad i = 1, 2, \dots, n$.

An arbitrary collection of events A_t where $t \in I$ for some index set I is mutually independent if every finite subcollection is mutually independent.

$n=2$, pairwise $\overset{\times}{\Rightarrow}$ Mutual

Definition 3: A finite collection of random variables $X_1, X_2, \dots, X_n$ is mutually independent if the collection

$$\{ X_i \in A_i \}_{i=1}^{n}$$ is mutually independent for all $A_i \in \mathcal{B}_{\mathbb{R}}$.

Proposition 1:- Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

X and Y are independent

if and only if

$$\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\, \mathbb{P}(Y \le y) \quad \text{for all } x, y \in \mathbb{R}. \qquad -(5)$$

Proof:-

$\boxed{\Rightarrow}$ let X and Y be independent

Suppose x and $y \in \mathbb{R}$. Then

$$(-\infty, x] \text{ and } (-\infty, y] \in \mathcal{B}_{\mathbb{R}}$$

$\Rightarrow$ $\mathbb{P}(X \in (-\infty, x], Y \in (-\infty, y])$

(Definition 3)

$$= \mathbb{P}(X \in (-\infty, x])\, \mathbb{P}(Y \in (-\infty, y])$$

$\Rightarrow$ $\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\, \mathbb{P}(Y \le y).$

$\boxed{\Leftarrow}$ We will need the following result:-

Proposition 2: Let Q_1, Q_2 be two Probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Suppose

$$Q_1((-\infty, x]) = Q_2((-\infty, x]) \quad \forall x \in \mathbb{R}$$

then $$Q_1(A) = Q_2(A) \quad \forall A \in \mathcal{B}_{\mathbb{R}}.$$

Suppose $\quad \mathbb{P}(X \le x) = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow \quad \mathbb{P}(X \in A) = 0 \quad \forall A \in \mathcal{B}_{\mathbb{R}}$

Ex: Using Proposition 2

$\Rightarrow \quad \mathbb{P}(X \in A, Y \in B) = 0 \quad \forall B \in \mathcal{B}_{\mathbb{R}}$

$\Rightarrow \quad \mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$

Suppose $\exists\, x_0 \in \mathbb{R}$ $\quad \mathbb{P}(X \le x_0) > 0.$

Define a Probability Q on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$Q(B) = \frac{\mathbb{P}(X \le x_0, Y \in B)}{\mathbb{P}(X \le x_0)} \quad B \in \mathcal{B}_{\mathbb{R}}. \qquad -(6)$$

Now $\Phi((-\infty, y]) = \mathbb{P}(Y \in (-\infty, y])$ by (5)

$\therefore$ $\Phi(B) \underset{\uparrow}{=} \mathbb{P}(Y \in B) \quad \forall B \in \mathcal{B}_{\mathbb{R}}$

(Proposition 2)

Applying above on (6) we have

$$\mathbb{P}(Y \in B)\, \mathbb{P}(X \leq x) = \mathbb{P}(Y \in B,\, X \leq x), \quad \forall x \in \mathbb{R}. \quad -(7)$$

(If $\mathbb{P}(X \leq x) = 0$ above is immediate)

Fix $B \in \mathcal{B}_{\mathbb{R}}$, s.t. $\mathbb{P}(Y \in B) > 0$

Define a Probability $\tilde{\Phi}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\tilde{\Phi}(A) = \frac{\mathbb{P}(X \in A,\, Y \in B)}{\mathbb{P}(Y \in B)} \quad \forall A \in \mathcal{B}_{\mathbb{R}} \quad -(8)$$

Using (7) $\Rightarrow$ and above

$$\tilde{\Phi}((-\infty, x]) = \mathbb{P}(X \leq x) \quad \forall x \in \mathbb{R}.$$

$$\Rightarrow \quad \tilde{\Phi}(A) = \mathbb{P}(X \in A) \quad \forall A \in \mathbb{R}.$$

Proposition 2

(8) $\Rightarrow$ $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$, $\forall B \in \mathcal{B}_{\mathbb{R}}$

(if $\mathbb{P}(Y \in B) = 0$ the above is immediate)

As $A, B \in \mathcal{B}_{\mathbb{R}}$ were arbitrary we are done. $\square$

4. **Events that occur infinitely often & almost all the time**

We have seen in Worksheet 1, Problem 3.

Example: Infinitely many coin tosses

$\Omega = \{0,1\}^{\mathbb{N}}$ $\mathcal{F} =$ σ-algebra generated by cylinder sets.

$A \in \mathcal{A}_n$ $\mathbb{P}(A) = \frac{|A|}{2^n}$.

H_n = event that the n^{th} toss is head.

[Events of interest]

Does H_n happen infinitely often?

i.e. Does $\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} H_n$ occur?

Do only finitely many tails occur?

ie. Does $\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} H_n$ occur?

Definition 1 :- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space.

Given $A_1, A_2, A_3, \ldots \in \mathcal{F}$ we define

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k := \{A_n \text{ occurs i.o.}\}$$

$$\liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k := \{A_n \text{ occurs a.a.}\}$$

Proposition 1:- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space

Given $A_1, A_2, A_3, \ldots \in \mathcal{F}$

$$\mathbb{P}(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mathbb{P}(A_n) \leq \limsup_{n\to\infty} \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_{n\to\infty} A_n)$$

Proof:- $\{\mathbb{P}(A_n)\}_{n\geq 1}$ is a sequence of real numbers. Therefore

$$\liminf_{n\to\infty} \mathbb{P}(A_n) \leq \limsup_{n\to\infty} \mathbb{P}(A_n) \qquad -(1)$$

is immediate.

Let $B_n = \bigcap_{k=n}^{\infty} A_k \quad \forall n\geq 1$

$$B_n \subseteq B_{n+1} \quad \forall n\geq 1$$

$\Rightarrow$ $\{\mathbb{P}(B_n)\}_{n\geq 1}$ is a increasing sequence

and

(by Problem 2 H.W. 1) $\mathbb{P}(B_n) \nearrow \mathbb{P}(B)$ as $n\to\infty$

with $B = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} A_k$

$$\therefore \mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k\right) \qquad -(2)$$

Now, $\mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k\right) \leq \mathbb{P}(A_n) \quad \forall n\geq 1$

$$\Rightarrow \quad \liminf_{n\to\infty} \mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k\right) \leq \liminf_{n\to\infty} \mathbb{P}(A_n) \qquad -(3)$$

From (2) and (3) we have

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mathbb{P}(A_n). \qquad -(4)$$

Similarly working with $C_n = \bigcup_{k=n}^{\infty} A_k$ and using $C_n \supseteq C_{n+1}$; Problem 1 Hw 1

One can show

$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) \geq \limsup_{n\to\infty} \mathbb{P}(A_n) \qquad -(5)$$

Using (5), (4), (1) we have the result. $\square$

Borel Cantelli lemma

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. Suppose we are given $A_1, A_2, A_3, \ldots \in \mathcal{F}$.

(i) If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0$

(ii) If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and $\{A_n\}_{n\geq 1}$ are independent then $\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0$

Remark 1:- Independence in (ii) is needed.

Take $A \in \mathcal{F}$ with $0 < \mathbb{P}(A) < 1$

let $A_n = A \quad \forall\, n \geq 1$

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = \mathbb{P}(A).$$

Proof:- (i) $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty \implies$

[Real Analysis Facts]:- $T_m = \sum_{k=m}^{\infty} \mathbb{P}(A_k) < \infty \quad \forall\, m \geq 1$ — (6)

and $T_m \to 0$ as $m \to \infty$

$$\limsup_{n\to\infty} A_n := \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k \subseteq \bigcup_{k=m}^{\infty} A_k \quad \forall\, m \geq 1$$

$$\implies \mathbb{P}(\limsup_{n\to\infty} A_n) \leq \mathbb{P}\left(\bigcup_{k=m}^{\infty} A_k\right)$$

(Ex: Use Facts (V) 4-8-2022) $\leq \sum_{k=m}^{\infty} \mathbb{P}(A_k) \quad \forall\, m \geq 1$

$$\implies 0 \leq \mathbb{P}(\limsup_{n\to\infty} A_n) \leq T_m \quad \forall\, m \geq 1$$

[Real Analysis Fact] Using (6),

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = 0$$

(ii) $$\left(\limsup_{n\to\infty} A_n\right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right)^c$$

$$= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c$$

$$\forall\, n \geq 1 \qquad \bigcap_{m=n}^{\infty} A_m^c \subseteq \bigcap_{m=n}^{n+k} A_m^c \quad \forall\, k \geq 1$$

$$\Rightarrow \quad \mathbb{P}\left(\bigcap_{m=n}^{\infty} A_m^c\right) \leq \mathbb{P}\left(\bigcap_{m=n}^{n+k} A_m^c\right) \quad \forall\, k \geq 1$$

[Independence of A_n $\Rightarrow$ Independence of A_n^c]

$$= \prod_{m=n}^{n+k} \mathbb{P}(A_m^c)$$

$$= \prod_{m=n}^{n+k} (1 - \mathbb{P}(A_m))$$

[$0 < x < 1 \Rightarrow 1 - x \leq e^{-x}$]

$$\leq e^{-\sum_{m=n}^{n+k} \mathbb{P}(A_m)} \qquad -(7)$$

[Real Analysis Fact]

As $\sum_{m=1}^{\infty} \mathbb{P}(A_m) = \infty$

$$\Rightarrow \forall\, n \geq 1 \quad T_k = \sum_{m=n}^{n+k} \mathbb{P}(A_m) \longrightarrow \infty \quad \text{as } k \to \infty$$

$-(8)$

As argued earlier from (7), (8) we have

$$\forall\, n \geq 1 \qquad \mathbb{P}\left(\bigcap_{m=n}^{\infty} A_m^c\right) = 0$$

$$\Rightarrow \quad \mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c\right) = 0$$

(Ex)

$\square$