Theorem 1 (Portmanteau): let P_n, $\mathbb{P}$ be probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. The following conditions are equivalent

(i) $\mathbb{P}_n \Rightarrow \mathbb{P}$

(ii) $\int f \, d\mathbb{P}_n \longrightarrow \int f \, d\mathbb{P}$ for all bounded uniformly continuous $f : \mathbb{R} \to \mathbb{R}$

(iii) $\limsup_{n \to \infty} \mathbb{P}_n(F) \leq \mathbb{P}(F)$ $\forall$ F-closed

(iv) $\liminf_{n \to \infty} \mathbb{P}_n(G) \geq \mathbb{P}(G)$ $\forall$ G-open

(v) $\lim_{n \to \infty} \mathbb{P}_n(A) = \mathbb{P}(A)$ for $A \in \mathcal{B}_{\mathbb{R}}$ s.t $\mathbb{P}(\partial A) = 0$.

Remark:- Suppose $\{X_n\}_{n \geq 1}$ are r.v. on $(\Omega_n, \mathcal{F}_n, \mathbb{Q}_n)$ and X is a r.v. on $(\Omega, \mathcal{F}, \mathbb{Q})$. We shall say

$$X_n \xrightarrow{d} X \quad \text{as } n \to \infty$$

if $\mathbb{Q}_n \circ X_n^{-1}(\cdot) := P_n(\cdot) \Longrightarrow P(\cdot) = \mathbb{Q} \circ X^{-1}(\cdot)$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv)

(iii) $\Leftrightarrow$ (v)

(i) $\Rightarrow$ (ii) is from definition of $\mathbb{P}_n \Rightarrow \mathbb{P}$.

(ii) $\Rightarrow$ (iii) let F be a closed set.

Let $\delta > 0$ be given

$x \in \mathbb{R}$, $\quad d(x,F) = \inf\{ |x-y| \;\; y \in F \}$

$$A_k = \{ x \in \mathbb{R} : d(x,F) < \tfrac{1}{k} \}$$

$$A_k \supseteq A_{k+1} \quad \& \quad \bigcap_{n=1}^{\infty} A_n = A$$

$\therefore$ $\exists\ \varepsilon > 0$ such that

$$\mathbb{P}(\{x : d(x,F) < \varepsilon\}) < \mathbb{P}(F) + \delta$$

$$\phi(t) = \begin{cases} 1 & \text{for } t \leq 0 \\ 1-t & 0 < t < 1 \\ 0 & t \geq 1 \end{cases}$$

let $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \phi\left(\frac{1}{\varepsilon} d(x,F) \right)$$

Ex. :- $f(x) \in [0,1]$

$f(x) = 1 \qquad x \in F$ $\qquad \boxed{2}$ $\qquad f$ is uniformly continuous.

$f(x) = 0 \qquad \text{if} \quad d(x, F) \geq \epsilon$

- $\mathbb{P}_n(F) = \int_{F_n} f \, d\mathbb{P}_n \leq \int f \, d\mathbb{P}_n$

- $\int f \, d\mathbb{P} = \int_{\{x : d(x,F) < \varepsilon\}} f \, d\mathbb{P} \leq \mathbb{P}(\{x : d(x,F) < \varepsilon\}) < \mathbb{P}(F) + \delta$

- $\int f \, d\mathbb{P}_n \longrightarrow \int f \, d\mathbb{P}$ as $n \to \infty$.

$\Rightarrow \quad \overline{\lim_{n \to \infty}} \, \mathbb{P}_n(F) \leq \mathbb{P}(F) + \delta$

As $\delta > 0$ was arbitrary $\Rightarrow$ (iii) holds

$\square$

(iii) $\Rightarrow$ (i) let f be a bounded continuous function. $\text{Range}(f) \subseteq [0,1]$

$\mathbb{Q}$ be a Probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Fix $k \geq 1$. $F_i = \{x \in \mathbb{R} : f(x) \geq \frac{i}{k}\}$, $1 \leq i \leq k$.

be closed sets.

$$\int f \, d\mathbb{Q} \geq \sum_{i=1}^{k} \frac{i-1}{k} \mathbb{Q}\left(\frac{i-1}{k} \leq f < \frac{i}{k}\right)$$

$$= \sum_{i=1}^{k} \frac{i-1}{k} \left[\mathbb{Q}(F_{i-1}) - \mathbb{Q}(F_i)\right]$$

$$= \frac{1}{k} \sum_{i=1}^{k} \mathbb{Q}(F_i) \quad -(1)$$

$$\int f \, d\mathbb{Q} \leq \sum_{i=1}^{k} \frac{i}{k} \left(\mathbb{Q}(F_{i-1}) - \mathbb{Q}(F_i)\right)$$

$$= \frac{1}{k} + \frac{1}{k} \sum_{i=1}^{k} \mathbb{Q}(F_i) \quad -(2)$$

$$\int f \, d\mathbb{P}_n \underset{(2)}{\leq} \frac{1}{k} + \frac{1}{k} \sum_{i=1}^{k} \mathbb{P}_n(F_i)$$

$$\int f \, d\mathbb{P} \underset{(1)}{\geq} \frac{1}{k} \sum_{i=1}^{k} \mathbb{P}(F_i)$$

Use (iii) to get

$$\limsup_{n\to\infty} \int f \, d\mathbb{P}_n \le \frac{1}{k} + \int f \, d\mathbb{P}$$

$k \geq 1$ was arbitrary $\Rightarrow$

$$\limsup_{n\to\infty} \int f \, d\mathbb{P}_n \le \int f \, d\mathbb{P}$$

Range $(f) \subseteq [-M, M]$

$g(x) = \alpha f(x) + \beta$ and $\alpha = \frac{1}{2M}$; $\beta = \frac{1}{2}$

Range $g \subseteq [0,1)$

$$\therefore \limsup_{n} \int g \, d\mathbb{P}_n \leq \int g \, d\mathbb{P}$$

$$\Rightarrow \limsup_{n\to\infty} \int f \, d\mathbb{P}_n \le \int f \, d\mathbb{P} \quad \text{—} (3)$$

Apply (3) to $h = -f$

$$\Rightarrow \liminf_{n\to\infty} \int f \, d\mathbb{P}_n \geq \int f \, d\mathbb{P} \quad \text{—} (4)$$

(3) and (4) $\Rightarrow$ $\int f \, d\mathbb{P}_n \longrightarrow \int f \, d\mathbb{P}$ as $n \to \infty$,

$\therefore$ (i) holds

(iii) $\Leftrightarrow$ (iv) follows by complimentation

(iii) $\Rightarrow$ (v) let $A^\circ :=$ interior of A

$\bar{A}$ = closure of A

$$\mathbb{P}(\bar{A}) \geq \overline{\lim_{n\to\infty}} \mathbb{P}_n(\bar{A}) \geq \overline{\lim_{n\to\infty}} \mathbb{P}_n(A)$$

$$\geq \underline{\lim_{n\to\infty}} \mathbb{P}_n(A) \geq \underline{\lim_{n\to\infty}} \mathbb{P}_n(A^\circ)$$

$$\geq \mathbb{P}(A^\circ) \quad \text{— (5)}$$

$\mathbb{P}(\partial A) = 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}_n(A) \overset{(5)}{=} \mathbb{P}(A)$.

$\bar{A} = A = A^\circ$

(v) $\Rightarrow$ (iii)

F be closed set

$F_\delta = \{x \in \mathbb{R} \mid d(x, F) \leq \delta\}$ $\forall \delta > 0$

$\partial F_\delta = \{x \in \mathbb{R} \mid d(x, F) = \delta\}$

EX: $\exists\ \delta_k \downarrow 0$ s.t. $\mathbb{P}(\partial F_{\delta_k}) = 0 \ \forall k \geq 1$

For $k \geq 1$

$$\overline{\lim_{n\to\infty}} \mathbb{P}_n(F) \leq \lim_{n\to\infty} \mathbb{P}_n(F_{\delta_k}) = \mathbb{P}(F_{\delta_k})$$

let $k \to \infty$ $\quad \overline{\lim_{n\to\infty}} \mathbb{P}_n(F) \leq \mathbb{P}(F)$

$\Rightarrow$ (iii) holds

$\square$

20 Characteristic functions

Characteristic functions are "Fourier transforms" of random variables.

Definition 1:- Let $X: \Omega \to \mathbb{R}$ be a random variable on Probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Its characteristic function $\phi_X : \mathbb{R} \to \mathbb{C}$ is given by

$$\phi_X(t) = E[e^{itX}] := E[\cos(tX)] + i\, E[\sin(tX)]$$

Remark 1:

- $\phi_X(\cdot)$ is well defined for all $t \in \mathbb{R}$
- We will denote $\phi_X(\cdot)$ by ϕ

- **Ex:-** Suppose $E[|X|^k] < \infty$ for some $k \geq 1$.

  Then for $0 \leq j \leq k$

  $$\phi_X^{(j)}(t) = E[(iX)^j e^{itX}] \quad \forall t \geq 0$$

  In particular, $\phi_X^{(j)}(0) = i^j E[X^j]$

- For a measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ we can define the characteristic function of μ as

$$\phi(t) \equiv \hat{\mu}(t) := \int_{\mathbb{R}} e^{itx} \mu(dx)$$

$$:= \int \cos(tx)\,\mu(dx) + i \int \sin(tx)\,\mu(dx)$$

If $\mu = \mathbb{P}.X^{-1}$ then $\phi \equiv \phi_X$.

Two objectives :-

- $\mathbb{P}.X^{-1} = \mathbb{P}.Y^{-1} \iff \phi_X(t) = \phi_Y(t) \quad \forall t \in \mathbb{R}$

(inversion) $\mu = \nu \iff \hat{\mu}(t) = \hat{\nu}(t)$

- $X_n \xrightarrow{d} X \iff \phi_{X_n}(t) \longrightarrow \phi_X(t) \quad \forall t \in \mathbb{R}$

(Continuity) $\mu_n \overset{w}{\Rightarrow} \mu \qquad \hat{\mu}_n(t) \longrightarrow \hat{\mu}(t) \quad \forall t \in \mathbb{R}$

Theorem 1 : (Fourier Inversion)

let μ be a Probability measure on $(\mathbb{R}, \mathcal{B}_{(\mathbb{R})})$ with characteristic function

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{itx} \mu(dx)$$

Then if $a<b$

$$\frac{1}{2}\mu(\{a\}) + \frac{1}{2}\mu(\{b\}) + \mu(a,b) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \hat{\mu}(t)\, dt$$

Proof :- For $T>0$,

$$\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\,\hat{\mu}(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\left(\int_{\mathbb{R}} e^{itx}\mu(dx)\right)dt$$

I $\overset{\text{Fubini}}{\underset{?}{=}} \frac{1}{2\pi}\int_{\mathbb{R}}\left[\int_{-T}^{T}\frac{e^{-it(a-x)}-e^{-it(b-x)}}{it}\,dt\right]\mu(dx)$

$$\underset{\checkmark}{\overset{\text{Calculus}}{=}} \frac{1}{2\pi}\int_{\mathbb{R}}\left[\int_{-T}^{T}\frac{\sin(t(x-a))-\sin(t(x-b))}{t}\,dt\right]\mu(dx) \quad -\text{(i)}$$

We will use the following lemma

$\boxed{\begin{array}{l}\textbf{lemma 1: } \theta\in\mathbb{R}, \\ \text{(a)}\ \displaystyle\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin(\theta t)}{t}\,dt = \pi\,\mathrm{Sign}(\theta) \\ \text{(b)}\ \exists\, M>0 \quad \displaystyle\sup_{T>0,\,\theta\in\mathbb{R}}\left|\int_{-T}^{T}\frac{\sin(\theta t)}{t}\,dt\right| \le M\end{array}}$

From (I),

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \hat{\mu}(t)\, dt$$

lemma 1

$$\underset{\text{Bounded Convergence Theorem}}{=} \frac{1}{2\pi} \int_{\mathbb{R}} [\pi \operatorname{Sign}(x-a) - \operatorname{Sign}(x-b)] \mu(dx)$$

$$= \frac{1}{2}\mu(\{a\}) + \mu((a,b)) + \frac{1}{2}\mu(\{b\})$$

$\square$

Proof of (I) : (Justification of Fubini)

$T>0$

$$\left| \frac{e^{-ita} - e^{-itb}}{it} e^{itx} \right| \leq \left| \int_a^b e^{-itr} dr \right| \leq \int_a^b |e^{-itr}| dr = b-a$$

$$\therefore \int_{\mathbb{R}} \int_{-T}^{T} \left| \frac{e^{-ita} - e^{-itb}}{it} e^{itx} \right| dt\ \mu(dx) \leq 2T(b-a)$$

– (2)

$$(t,x) \longrightarrow \frac{e^{-ita} - e^{-itb}}{it} e^{itx}$$ is jointly measurable on $\mathbb{R}_+ \times \mathbb{R}$

– (3)

②, ③ $\Rightarrow$ Fubini's Theorem applies. $\square$

Proof of lemma 1 :-

$T > 0$

$\theta = 0.$ $\Rightarrow$ $\int_{-T}^{T} \frac{\sin(\theta t)}{t}\, dt = 0$

$\therefore \lim_{T\to\infty} \int_{-T}^{T} \frac{\sin(\theta t)}{t}\, dt = 0 = \text{Sign}(0) = \text{Sign}(\theta)$ — ④

$\theta \neq 0$

$$\int_{-T}^{T} \frac{\sin(\theta t)}{t}\, dt \overset{s=|\theta|t}{=} \text{Sign}(\theta) \int_{-|\theta|T}^{|\theta|T} \frac{\sin(s)}{s}\, ds$$

$$= 2\,\text{Sign}(\theta) \int_{0}^{|\theta|T} \frac{\sin(s)}{s}\, ds \quad \text{— ⑤}$$

Ex:-

(i) $\int_{0}^{|\theta|T} \frac{\sin(s)}{s} du = \int_{0}^{\infty} \left[\int_{0}^{|\theta|T} \sin(s)\, e^{-us}\, ds \right] du$ (Fubini)

(ii) $\int_{0}^{|\theta|T} \sin(s)\, e^{-us}\, ds = \frac{1 + h(\theta,T) - u\, g(\theta,T)}{1+u^2}$

(Integration by parts)

where $\quad g(\theta,T) \to 0$
$\quad h(\theta,T) \to 0 \quad$ as $T \to \infty$.

(iii) $$\lim_{T\to\infty} \int_0^{|\theta|T} \frac{\sin(s)}{s}\, ds = \int_0^{\infty} \frac{1}{1+u^2}\, du = \frac{\pi}{2}$$

(Dominated Convergence)

Part (a) follows from (iii), (b) follows from (5) □

We are now ready to resolve the first objective)

Corollary 1 (Fourier Uniqueness – "characteristic")

let X, Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.
Then

$$\phi_X(t) = \phi_Y(t) \quad \forall\ t \in \mathbb{R}$$

$$\iff \quad \mathbb{P} \circ X^{-1}(\cdot) = \mathbb{P} \circ Y^{-1}(\cdot)$$

Proof:-

$\Leftarrow$ obvious.

$\Rightarrow$ Theorem 1 $\Rightarrow$ $a<b$

$$\frac{\mathbb{P}\cdot X^{-1}(\{a\}) + \mathbb{P}\cdot X^{-1}(\{b\})}{2} + \mathbb{P}\cdot X^{-1}((a,b)) = \frac{\mathbb{P}\cdot Y^{-1}(\{a\}) + \mathbb{P}\cdot Y^{-1}(\{b\})}{2} + \mathbb{P}\cdot Y^{-1}((a,b))$$

$\Rightarrow$ outside a countable collection of $\{\alpha_n\}_{n\geq 1}$

$a, b \notin \{\alpha_n\}$

$$\mathbb{P}\cdot X^{-1}(\{a\}) = \mathbb{P}\cdot X^{-1}(\{b\}) = \mathbb{P}\cdot Y^{-1}(\{a\}) = \mathbb{P}\cdot Y^{-1}(\{b\}) = 0$$

$$\Rightarrow \quad \mathbb{P}\cdot X^{-1}((a,b]) = \mathbb{P}\cdot Y^{-1}((a,b])$$

for $a<b$ such $b, a \notin \{\alpha_n\}_{n\geq 1}$

$$\overset{\text{Ex}}{\Rightarrow} \quad \mathbb{P}\cdot X^{-1} = \mathbb{P}\cdot Y^{-1} \qquad \square$$