18 Three Series Theorem

In Probability / Applications one is always interested in understanding how

$$S_n = \sum_{i=1}^{n} X_i \quad \text{behaves as } n \to \infty,$$

where $\{X_i\}_{i \geq 1}$ are i.i.d. r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$.

Example:-

- $X_i \sim \text{Bernoulli}(p)$, then S_n is the number of heads in n-tosses of a coin with probability of heads $= p$; $0 \leq p \leq 1$.

- $X_i = \begin{cases} +1 & \text{w.p } p \\ -1 & \text{w.p } 1-p \end{cases}$ Then S_n is the position of a random walk on $\mathbb{Z}$.

- $X_i \geq 0$ (general distribution) then S_n is called a renewal process

A famous theorem that deals with sums is due to Kolmogorov. It provides necessary and sufficient conditions for S_n to converge.

They are given in terms of three other series. One assumes that $\{|X_n| > 1\}$ occurs finitely often, the second assumes that $E\{X_n \mathbb{1}_{|X_n| \leq 1}\}$ sums up and the third assumes that the variances of $X_n \mathbb{1}_{|X_n| \leq 1}$ sum up

Theorem 1 (Kolmogorov three series theorem): If $\{X_n\}_{n\geq 1}$ are independent random variables then

$$S_n = \sum_{k=1}^{n} X_k \quad \text{Converges almost surely}$$

iff

(i) $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > 1) < \infty$

(ii) $\sum_{n=1}^{\infty} E[X_n \mathbb{1}_{|X_n| \leq 1}] < \infty$

(iii) $\sum_{n=1}^{\infty} E[(X_n \mathbb{1}_{|X_n| \leq 1} - E[X_n \mathbb{1}_{|X_n| \leq 1}])^2] < \infty$

Equivalently

$Y_j = X_j \mathbb{1}_{|X_j| \leq 1}$

- $\sum_{n=1}^{\infty} \mathbb{P}(X_j \neq Y_j) < \infty$
- $\sum_{n=1}^{\infty} E[Y_n] < \infty$
- $\sum_{n=1}^{\infty} \mathrm{var}[Y_n] < \infty$

Proof:-

- $\boxed{\text{Sufficiency : (i), (ii), (iii)} \Rightarrow \{S_n\}_{n\geq 1} \text{ Converges a.s.}}$

For this we will need the following result:-

Theorem 2 (Khintchine - Kolmogorov) let $\{Y_n\}_{n\geq 1}$ be independent r.v's such that

$$E[Y_n] = 0 \quad \forall n \geq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} E[Y_n^2] < \infty$$

Then there exist r.v. Y such that

$$\sum_{k=1}^{n} Y_k \xrightarrow{a.s.} Y \quad \text{as } n \to \infty.$$

Assuming the result we shall finish the proof of sufficiency (Theorem 1).

let $Y_n = X_n \mathbb{1}_{|X_n| \leq 1} - E[X_n \mathbb{1}_{|X_n| \leq 1}] \quad \forall n \geq 1.$

Then Y_n's are independent and $E[Y_n] = 0.$

Further $\sum_{n=1}^{\infty} E[Y_n^2] < \infty$ by (iii)

Theorem 2 $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} Y_n < \infty$ a.s.

Using (ii) $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} X_n \mathbb{1}_{|X_n| \le 1} < \infty$ a.s. — (1)

Now let $Z_n = X_n \mathbb{1}_{|X_n| \le 1}$

$$\mathbb{P}(X_n \neq Z_n) = \mathbb{P}(|X_n| > 1)$$

By (i) $\displaystyle\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq Z_n) < \infty$

Borel $\Rightarrow$ Cantelli $\mathbb{P}(X_n \neq Z_n \text{ i.o.}) = 0$ — (2)

$\therefore$ (1)

$\Rightarrow$ $\mathbb{P}\left(\displaystyle\sum_{n=1}^{\infty} Z_n < \infty\right) = 1$

& (2)

$\Rightarrow$ $\mathbb{P}\left(\displaystyle\sum_{n=1}^{\infty} X_n < \infty\right) = 1$

$\Rightarrow$ $\mathbb{P}(S_n \text{ converges}) = 1$

$\square$

- **Necessary:** S_n converges a.s to X then (i), (ii) & (iii) hold.

We will need the following lemma for proving the necessary direction.

(Two Series Theorem)

Lemma 3 :- If Y_n's are independent uniformly bounded r.v.'s i.e $\exists\, c > 0$ $\mathbb{P}(|Y_n| \le c \;\forall n \ge 1) = 1$ &

- $T_n = \displaystyle\sum_{j=1}^{n} Y_j$ converges a.s. to T as $n \to \infty$

Then – $\displaystyle\sum_{n=1}^{\infty} E[Y_n] < \infty$ and $\displaystyle\sum_{n=1}^{\infty} \operatorname{Var}(X_n) < \infty$.

Assuming the result we shall finish the proof of necessary part. (Theorem 1).

As $S_n = \sum_{k=1}^{n} X_n$ Converges to S a.s as $n \to \infty$

$$\Rightarrow \quad X_n \xrightarrow{a.s} 0 \quad \text{as} \quad n \to \infty$$

$$\Rightarrow \quad \mathbb{P}\left(\limsup_{n \to \infty} |X_n| > 1 \right) = 0$$

Borel - Cantelli

Xn's are independent

$$\Rightarrow \quad \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > 1) < \infty \quad \Rightarrow \quad \text{(i) holds} \quad - (3)$$

If $Y_n = X_n \mathbb{1}_{|X_n| \le 1}$ then we have

$$\mathbb{P}(Y_n \neq X_n) = \mathbb{P}(|X_n| > 1)$$

Using (3) $\sum_{n=1}^{\infty} \mathbb{P}(Y_n \neq X_n) < \infty$

Borel - Cantelli

$$\Rightarrow \quad \mathbb{P}(Y_n \neq X_n \ \text{i.o.}) = 0 \quad - (4)$$

Using (4) & $S_n = \sum_{k=1}^{n} X_n$ Converges a.s. we have

$$\sum_{k=1}^{n} Y_n \ \text{Converge a.s} \quad - (5)$$

As Y_n's are independent and uniformly bounded by 1 and (5) by lemma 3 we have

$$\sum_{n=1}^{\infty} Y_n^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \text{var}(Y_n) < \infty$$

which implies that (ii) and (iii) holds □

We will now prove Theorem 2 and Lemma 3.

One Series Theorem

Theorem 2 (Khintchine - Kolmogorov) let $\{Y_n\}_{n\geq 1}$ be independent r.v.'s such that

$$E[Y_n] = 0 \quad \forall n \geq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} E[Y_n^2] < \infty$$

Then there exist r.v. Y such that

with probability one (i) $T_n = \sum_{k=1}^{n} Y_k$, $T_n \xrightarrow{a.s.} Y$ as $n \to \infty$,

(in quadratic mean) (ii) $E[(T_n - Y)^2] \longrightarrow 0$ as $n \to \infty$ and

(iii) $E[Y^2] = \sum_{k=1}^{\infty} E[Y_k^2] < \infty$

Extra not required for proof of Theorem 1

Proof: Now: $T_m - T_n = \sum_{k=n+1}^{m} Y_k \quad m \geq n$

let $\varepsilon > 0$ be given

$$\mathbb{P}(|T_n - T_m| > \varepsilon) \leq \frac{1}{\varepsilon^2} E[(T_n - T_m)^2]$$

$$= \frac{1}{\varepsilon^2} E\left(\sum_{k=n+1}^{m} Y_k\right)^2$$

independence & $\forall k \geq 1$ $E[Y_k] = 0$

$$= \frac{1}{\varepsilon^2} \sum_{k=n+1}^{m} E(Y_k)^2$$

By hypothesis that $\sum_{k=1}^{\infty} E Y_k^2 < \infty$

$$\Rightarrow \sup_{m \geq n} \mathbb{P}(|T_n - T_m| > \varepsilon) \longrightarrow 0 \quad \text{as } n \to \infty$$

Proposition 17.1 (i) $\Rightarrow$ $\exists\, Y$ r.v such that

$$T_n \xrightarrow{p} Y \quad \text{as } n \to \infty.$$

By Theorem 17.1 (Levy's Theorem)

$$T_m \xrightarrow{a.s} Y \quad \text{as } m \to \infty$$

(i) is done.

Now by Fatou's lemma

$$0 \le E|T_k - Y|^2 \le \varliminf_{m \to \infty} E|T_k - T_m|^2$$

$$\overset{\text{independence}}{\le} \sum_{l=k}^{\infty} E[Y_l^2] \qquad \forall\, m \ge 1$$

As $\sum_{k=1}^{\infty} E[Y_k^2] < \infty \Rightarrow$

(ii) is done

$$E[T_k - Y]^2 \longrightarrow 0 \quad \text{as } m \to \infty.$$

— (6)

(iii) let $\varepsilon > 0$ be given.

$A \in \mathcal{F}$ $\quad E[T_m^2 1_A] \leq 4[\ E[Y^2 1_A] + E[(T_m - Y)^2 1_A]]$

- $\exists\ N \geq 1$ st $m \geq N$ $\quad$ by (6)

$$E[(T_m - Y)^2 1_A] \leq E[(T_m - Y)^2] < \varepsilon$$

- $\exists\ \delta > 0$ st $P(A) < \delta \implies$

$$\max\left\{ E[Y^2 1_A],\ \max_{m \leq N} E[(T_m - Y)^2 1_A] \right\} < \varepsilon$$

$\implies \forall\ m \geq 1$ $\quad P(A) < \delta \implies$

$$E[T_m^2 1_A] < 4[2\varepsilon] = 8\varepsilon$$

$\implies \{T_m^2\}_{m \geq 1}$ are uniformly integrable.

By Theorem 15.3 : $\quad E[T_m^2] \longrightarrow E[Y^2]$ as $m \to \infty$

$$\implies E[Y^2] = \lim_{k \to \infty} \sum_{m=1}^{k} E[Y_m^2]$$

(iii) done

$\square$

(2-Series Theorem)

lemma 3 :- If Y_n's are independent uniformly bounded r.v's i.e $\exists\ c > 0$ $P(|Y_n| \leq c\ \forall\ n \geq 1) = 1$ &

- $T_n = \sum_{j=1}^{n} Y_j$ converges a.s. to T as $n \to \infty$

Then - $\sum_{n=1}^{\infty} E[Y_n] < \infty$ and $\sum_{n=1}^{\infty} \mathrm{var}(Y_n) < \infty$.

Proof :- let $k > 0$ such that

(Ex.) $\quad P(\sup_{n \geq 1} |T_n| < k) > 0$

$S : \Omega \to \mathbb{R}_+ \cup \{\infty\}$

$$S = \inf\{n \geq 1 \mid |T_n| \geq k\} \qquad \mathbb{P}(S=\infty) > 0$$

$$U_n = T_{\min\{S,n\}} = \sum_{j=1}^{n} Y_j \, 1_{\{S \geq j\}}$$

As $n \to \infty$, $\quad U_n \xrightarrow{a.s} T_S$

$$\sup_n |U_n| = \sup_n |T_{\min(S-1,n-1)} + Y_{\min(S,n)}|$$

$$\leq k + C$$

$$\Rightarrow \quad E \sup_n |U_n| < \infty$$

D.C.T.

$$\Rightarrow \quad E[U_n] \longrightarrow E\, T_S \quad \text{as } n \to \infty$$

Now $\quad E[U_n] = \sum_{j=1}^{n} \mathbb{P}(S \geq j)\, E[Y_j]$

$\{|T_m| < k : m < j\} = \{S \geq j\} \in \sigma(Y_1, Y_2, \dots, Y_{j-1})$

$$\Rightarrow \quad E[Y_n] = \frac{E[U_n] - E[U_{n-1}]}{\mathbb{P}(S \geq n)}$$

$$b_n = \frac{1}{\mathbb{P}(S \geq n)} \qquad a_n = E[U_n] - E[U_{n-1}], \quad a_0 = A_0 = 0$$

$$A_n = \sum_{j=0}^{n} a_j$$

$$\sum_{j=1}^{n} E[Y_j] = \frac{E[U_n]}{\mathbb{P}(S \geq n)} - \sum_{j=1}^{n-1} \left[\frac{1}{\mathbb{P}(T \geq j+1)} - \frac{1}{\mathbb{P}(T \geq j)} \right] E[U_j]$$

is of the form

$$\sum_{j=1}^{n} a_j b_j = A_n b_n - A_0 b_1 - \sum_{j=1}^{n-1} (b_{j+1} - b_j) A_j$$

$\sum a_n < \infty$
b_n bounded $\quad \Rightarrow \quad \sum_{j=1}^{\infty} a_j b_j < \infty$

$$\Rightarrow \quad \sum_{j=1}^{\infty} E[Y_j] < \infty \quad \text{as} \quad \begin{array}{l} E[U_n] \to E[U_S] \\ \mathbb{P}(S=\infty) > 0 \end{array}$$

Now look at $L_n = \sum_{i=1}^{n} (Y_i - E[Y_i])$, $L_n \xrightarrow{a.s} L$.

Choose $K_1 > 0$ s.t. $\mathbb{P}(\sup_n |L_n| < K_1) > 0$

$$V = \inf\{n \geq 1 \mid |L_n| \geq K_1\}$$

Note: $\mathbb{P}(V = \infty) > 0$

$$V_n = \sum_{j=1}^{n} (Y_j - E[Y_j]) \mathbb{1}(V \geq j)$$

$$V_n^2 = V_{n-1}^2 + 2V_{n-1}(Y_n - E[Y_n]) \mathbb{1}_{V \geq n} + (Y_n - E[Y_n])^2 \mathbb{1}_{V \geq n}$$

$$E V_n^2 = E V_{n-1}^2 + \mathbb{P}(V \geq n) \operatorname{Var}(Y_n)$$

$$\Rightarrow \quad E V_n^2 \leq C_1$$

$$\Rightarrow \quad \mathbb{P}(V = \infty) \sum_{j=1}^{n} \operatorname{Var}(Y_j) \leq E[V_n^2] \leq C_1$$

$$\Rightarrow \quad \sum_{j=1}^{\infty} \operatorname{Var}(Y_j) < \infty \qquad \square$$

19 Weak-Convergence

$(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. $\{X_n : n \geq 1\}$ and X be random variables on it.

- $X_n \xrightarrow{p} X$ as $n \to \infty$ [Convergence in Probability]
- $X_n \xrightarrow{a.s.} X$ as $n \to \infty$ [Convergence almost surely].

[Proposition 1 and 2 in 17]

Example :- $X_n \sim \text{Binomial}(n, \frac{\lambda}{n})$

- $$\Phi_n(\{k\}) = \mathbb{P}(X_n = k) \xrightarrow[n\to\infty]{} \underbrace{\frac{e^{-\lambda}\lambda^k}{k!}}_{} \quad \forall k = 0, 1, 2, \ldots \quad -(1)$$

$$X_n \sim \text{Binomial}(n, \tfrac{\lambda}{n}) \xrightarrow[n\to\infty]{} X \sim \text{Poisson}(\lambda)$$

$$(1) \iff \mathbb{P}(X_n \leq x) \xrightarrow[n\to\infty]{} \mathbb{P}(X \leq x) \quad x \in \ldots$$

- $X_n \sim \text{Binomial}(n, p)$

$$Z_n((a,b]) = \mathbb{P}\left(a < \frac{X_n - np}{\sqrt{np(1-p)}} \leq b\right)$$

Binomial C.L.T (De-Moivre)

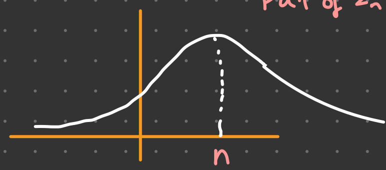
$$\xrightarrow[n\to\infty]{} \int_a^b \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx \quad -(2)$$

$$\parallel$$

$$Z((a,b])$$

② $\sqrt{np(1-p)}$ .. Scale $\quad$ Binomial (n,p) $\xrightarrow[n\to\infty]{}$ Normal $(0,1)$.

np .. Centre



- $Z_n \sim \text{Normal}(n, 1)$

$\xrightarrow[n\to\infty]{}$ " mass Escapes to ∞ ".

Definition 1 :- Given a sequence of probability measures $\{\mathbb{P}_n(\cdot)\}_{n\geq 1}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ we shall say

" $\mathbb{P}_n(\cdot)$ — Converges weakly to a measure $\mathbb{P}$ " as $n\to\infty$

$\boxed{\text{denoted by } \quad \mathbb{P}_n \Rightarrow \mathbb{P}}$

if $\displaystyle\int_{\mathbb{R}} f\, d\mathbb{P}_n \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} f\, d\mathbb{P}$ for all bounded

Continuous $f: \mathbb{R} \to \mathbb{R}$

Remark1 :

X_n ... law is given by $\mathbb{P}_n$ i.e. $\mathbb{P}(X_n \in \cdot) \equiv \mathbb{P}_n(\cdot)$

X ... $\mathbb{P}$ i.e. $\mathbb{P}(X\in \cdot) \equiv \mathbb{P}(\cdot)$

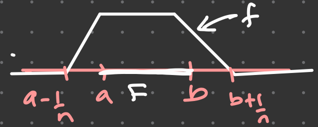
" $X_n \xrightarrow[n\to\infty]{\text{weakly}} X$ " : $\mathbb{E}[f(X_n)] := \int f\, d\mathbb{P}_n \longrightarrow \int f\, d\mathbb{P} := \mathbb{E}[f(X)]$

f — bounded continuous f

$\mathbb{P}_n \Rightarrow \mathbb{P}$ as $n\to\infty$

- "weak Convergence" .. has connections $\underline{\text{weak * topology}}$

- "bounded continuous functions" - captures the topology of $\mathbb{R}$. i.e. "F- closed set" $1_F(\cdot) \rightsquigarrow f$ bounded continuous



- well - defined notion : $\mathbb{P}_n \Rightarrow \mathbb{P}$ and $\mathbb{P}_n \Rightarrow \mathbb{Q}$ then $\mathbb{Q} = \mathbb{P}$ (True : $\underbrace{}_{\text{force}}$ $\int f d\mathbb{P} = \int f d\mathbb{Q}$ $\forall f$ - bounded continuous)

Theorem 1 (Portmanteau) :: let $\{\mathbb{P}_n\}_{n\geq 1}$, $\mathbb{P}$ be Probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. The following conditions are equivalent :-

(i) $\mathbb{P}_n \Rightarrow \mathbb{P}$

(ii) $\int f d\mathbb{P}_n \longrightarrow \int f d\mathbb{P}$ for all bounded uniformly continuous $f : \mathbb{R} \to \mathbb{R}$

(iii) $\limsup_{n\to\infty} \mathbb{P}_n(F) \leq \mathbb{P}(F)$ $\forall F$ - closed

(iv) $\liminf_{n\to\infty} \mathbb{P}_n(G) \geq \mathbb{P}(G)$ $\forall G$ - open

(v) $A \in \mathcal{B}_{\mathbb{R}}$ $\mathbb{P}(\partial A) = 0$ then

$\mathbb{P}_n(A) \longrightarrow \mathbb{P}(A)$ as $n \to \infty$.

Remark 2:

- $X_n \sim \mathbb{P}_n$ and $X \sim \mathbb{P}$

"$X_n \xrightarrow[n\to\infty]{\text{weakly}} X$" $\iff$ $\mathbb{P}_n \Rightarrow \mathbb{P}$

$\overset{(v)}{\iff}$ $F_n(x) := \mathbb{P}(X_n \leq x) = \mathbb{P}(X_n \in (-\infty, x])$

$\xrightarrow[n\to\infty]{}$ $\mathbb{P}(X \leq x) := F(x)$ x is a continuity point of $F(\cdot)$

Provided $\mathbb{P}(\partial(-\infty, x]) = 0$ $\iff$ $\mathbb{P}(\{x\}) = 0$ $\iff$

(Alternative: $X_n \xrightarrow{d} X$ as $n \to \infty$)

- $\mathbb{P}_n(dx) = \text{Uniform}\{\frac{i}{n} : 1 \leq i \leq n\}$

f - bounded continuous

$\int f \, d\mathbb{P}_n = \sum_{i=1}^{n} \frac{1}{n} f(\frac{i}{n}) \xrightarrow[n\to\infty]{\text{Riemann sums}} \int_0^1 f(x)\,dx$

- $\mathbb{P}(dx) \equiv \text{Uniform}(0,1)$ $\int f \, d\mathbb{P} = \int_0^1 f(x)\,dx$

$$\boxed{\text{So } \quad \mathbb{P}_n \Rightarrow \mathbb{P}}$$

(v) - implications

$A = \mathbb{Q} \cap [0,1]$

$$\mathbb{P}_n(A) = 1 \quad \& \quad \mathbb{P}(A) = 0$$

(v) - not violated as $\mathbb{P}(\partial A) = \mathbb{P}([0,1]) = 1.$

- $\{x_n\}_{n \geq 1}$ be a sequence in $\mathbb{R}$. such that

$$x_n \longrightarrow x \quad \text{as} \quad n \to \infty.$$

$$\mathbb{P}_n(\cdot) = \delta_{x_n}(\cdot) \quad ; \quad \mathbb{P}(\cdot) = \delta_x(\cdot)$$

Verify Theorem 1 (i) ... (v) :-

(i) f is bounded continuous function. : $\int f \, d\mathbb{P}_n = f(x_n)$

$(\mathbb{P}_n \Rightarrow \mathbb{P})$

$$\int f \, d\mathbb{P} = f(x) \xleftarrow{n \to \infty} f(x_n)$$

(ii) f is bounded uniformly continuous (True from (i))

(iii) F - closed set

$x \in F \Rightarrow \mathbb{P}(F) = 1$ and clearly

$$\overline{\lim_{n \to \infty}} \mathbb{P}_n(F) \leq 1 = \mathbb{P}(F) \quad ✓$$

$x \notin F \Rightarrow \mathbb{P}(F) = 0$

F^c – open $\equiv$ $\exists N: n \geq N \quad x_n \notin F.$

$x \in F^c$

$\exists \varepsilon > 0 \quad B(x, \varepsilon) \subseteq F$ $\Rightarrow \mathbb{P}_n(F) = 0 \quad \forall n \geq N$

$$\Rightarrow \overline{\lim_{n \to \infty}} \mathbb{P}_n(F) \leq 0 = \mathbb{P}(F) \quad ✓$$

(iv) Take Compliments & use (iii)

(v) $A \in \mathcal{B}_{\mathbb{R}} \quad \mathbb{P}(\partial A) = 0 \iff x \notin \partial A$

$\iff \exists \varepsilon_0 > 0$ s.t $(x - \varepsilon_0, x + \varepsilon_0) \subseteq A$ or

$(x - \varepsilon_0, x + \varepsilon_0) \subseteq A^c$

$\Rightarrow \exists N \geq 1$ st. $n \geq n$

$x_n \in (x - \varepsilon_0, x + \varepsilon_0)$ $\subseteq A$ or $\subseteq A^c$

$\Rightarrow \exists N \geq 1$ st. $n \geq n$

$\mathbb{P}_n(A) = 1 = \mathbb{P}(A)$ $\therefore$ (v) holds

or

$\mathbb{P}_n(A) = 0 = \mathbb{P}(A)$

- $\{x_n\}_{n \geq 1}$ differ from x

(iii) $F = \{x\}$ – inequality is strict. (v) $A = \{x_{n_k} : k \geq 1\}$

$\mathbb{P}_n(A)$ has no limit