

Probability Theory [Second Course in Measure Theoretic Probability]

- (I)
- Probability Theory [as seen in undergraduate classes]
 - Measure Theory - Review

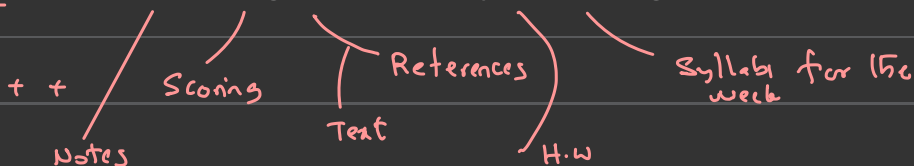
(II) Build on the machine established in (I).

Basics :- - H.W. due every week [Moodle]

[Question] - Quiz every week (Thursday)
from
the H.W.

- Midterm / Final (Straight forward)

Website : www.isibang.ac.in/~athreya/Teaching/c12



Question / Direction of study :-

Q - Need for Measure Theory ?

Measure Theory :-

Sample Space

Ω := any non empty set

[Probability] :- Set of all possible outcomes of an experiment

$\mathcal{F} \subseteq \mathcal{P}(\Omega)$:= Power set of Ω .

(i) $\phi, \Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_n \in \mathcal{F} \forall n \geq 1$ then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$$

[σ -algebra]
[σ -field]

[$\infty \rightarrow$ by $m \neq l$
algebra or field]

[Probability] :- Event (subcollection of outcomes)

$$\mathbb{P}: \mathcal{F} \rightarrow [0, 1]$$

(1) $\mathbb{P}(\Omega) = 1$

(2) $E_n \in \mathcal{F}, E_n \cap E_l = \phi$ for $m \neq l$

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$$

[Probability] :- function on sets satisfying Axioms (1) and (2)]

Examples:-

Experiment 1:- Toss a ^(fair) coin and let

$$X = \begin{cases} 1 & \text{if Head appears} \\ 0 & \text{if Tail appears} \end{cases}$$

$$\Omega = \{H, T\}$$

$$\mathcal{F} = \mathcal{P}(\Omega) := \{ \emptyset, \Omega, \{H\}, \{T\} \}$$

[Model] $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ (Probability)

fair

$$\mathbb{P}(\{H\}) = \frac{1}{2} = \mathbb{P}(\{T\})$$

$$X : \Omega \rightarrow \{0, 1\} = \mathbb{T}$$

$$\mathcal{F}' = \{ \emptyset, \mathbb{T}, \{0\}, \{1\} \}$$

$$\mathbb{Q}(\{1\}) = \mathbb{P}(X=1) = \frac{1}{2}$$

$$\mathbb{Q}(\{0\}) = \mathbb{P}(X=0) = \frac{1}{2}$$

$$(\Omega, \mathcal{F}, \mathbb{P})$$

X

$$(\mathbb{T}, \mathcal{F}', \mathbb{Q})$$

random variable

Experiment 2: Let $n \geq 1$ be given.

Toss a ^{fair} coin say n -times.

[Model]

$$\Omega = \{H, T\}^n$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$\mathbb{P}: \mathcal{F} \rightarrow [0, 1]$$

... (?)

$X_n =$ Counts the # of Heads in n -tosses

$$\mathcal{T} = \{0, 1, 2, \dots, n\}$$

$$\mathcal{F}' = (?)$$

$$\mathbb{Q}: \mathcal{F}' \rightarrow [0, 1]$$

sb-w: $\mathbb{Q}_n(\{k\}) = \binom{n}{k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2}\right)^k$

$$k=0, 1, 2, \dots, n$$

Binomial
distribution

Theorem - P :- Coin with (biased) probability of
heads $= p/n$

$$\mathbb{Q}_n(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k} \xrightarrow{n \rightarrow \infty} \frac{e^{-p} p^k}{k!}$$

(Toss a coin n times)
bias $= p/n$

$$\forall k \equiv 0, 1, 2, \dots$$

Poisson (p) :-
Distribution

$$\Omega = \mathbb{N} \cup \{0\}, \quad \mathcal{F} = \mathcal{P}(\mathbb{N} \cup \{0\})$$

$$\mathbb{P}_p : \mathcal{F} \rightarrow [0, 1]$$

$$\text{by } \mathbb{P}_p(\{k\}) := \frac{e^{-p} p^k}{k!} \quad \forall k \in \mathbb{N}$$

$$\left[A \in \mathcal{F} \right] \quad \mathbb{P}_p(A) := \sum_{k \in A} \mathbb{P}_p(\{k\})$$

$\mathbb{P}(X=k) \equiv X$ - Poisson Random variable

Theorem (C.L.T.) :- Toss a coin n times
(fair)

$X_n = \#$ of Heads in n -tosses

$$\mathbb{P}\left(\frac{a - \frac{n}{2}}{\sqrt{\frac{n}{4}}} < X_n \leq \frac{b - \frac{n}{2}}{\sqrt{\frac{n}{4}}}\right) \xrightarrow{n \rightarrow \infty} \int_a^b \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

(Normal distribution)

$$\Omega = \mathbb{R}$$

$\mathcal{F} = \mathcal{B}(\mathbb{R})$ - Borel σ -algebra on \mathbb{R}

$\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ - [Probability]

- Exists \mathbb{P}
- Unique!

$$\mathbb{P}((a, b]) = \int_a^b \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

$$\mathbb{P}(X \in (a, b])$$

X - Normal random variable

Undergraduate Probability - classification

[Discrete Random Variables]

X : Bernoulli (p)

Binomial (p)

Poisson (λ)

[Continuous Random Variables]

X : Normal ($0, 1$) , Uniform ($0, 1$)

Define: (Neither discrete nor continuous)
Toss a coin (fair)

if heads then $W \stackrel{d}{=} \text{Poisson}(\lambda)$

if tails then $W \stackrel{d}{=} \text{Normal}(0, 1)$

Ex: (Ω, \mathcal{F}, P) that will specify distribution of W .

First Motivation: - Set up Abstract Measure Theory.

Example: $X :=$ uniform chosen number in $[0,1]$

$\mathcal{A} = [0,1]$ $\mathcal{F} = ?$ $\mathbb{P}: \mathcal{F} \rightarrow [0,1]$

• $\mathbb{P}([a,b]) = b-a$ $0 \leq a \leq b \leq 1$ - (1)

(Proportional to length of interval)

• A, B to be two disjoint intervals

[finite additivity]

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ - (2)

• $\{E_n\}_{n \in \mathbb{N}}$ $E_n \cap E_k = \emptyset$ then

[countable additivity]

$\mathbb{P}\left(\bigcup_{m=1}^{\infty} E_m\right) = \sum_{m=1}^{\infty} \mathbb{P}(E_m)$ - (3)

well defined event

Series converges.

Questions :-

(i) \mathbb{P} - have uncountable additivity?

(ii) Can there exists a \mathbb{P} on $\mathcal{P}(\mathcal{A})$

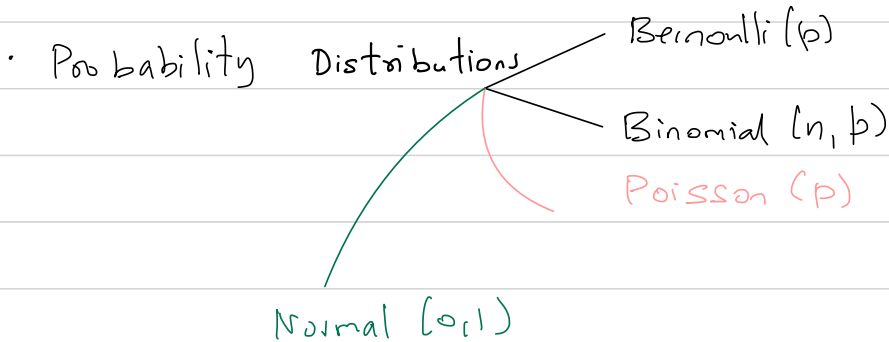
St \longrightarrow (1), (2)

\longrightarrow (1), (2), (3)

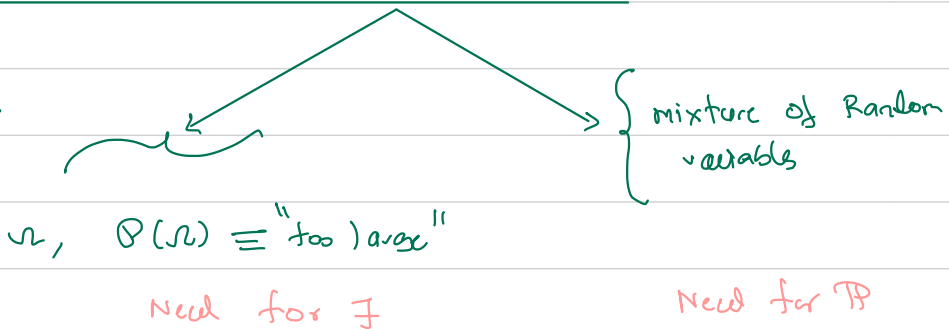
Recall:

Q - need for Measure Theory

- why are each of the properties required ?



Need for Abstract Measure Theory



4-8-2022 :-

Ω - non-empty set

\mathcal{F} - σ -algebra $\equiv \begin{cases} \Omega, \phi \in \mathcal{F} \\ \text{- closed under complement} \\ \text{- closed under countable unions} \end{cases}$

$\mathbb{P}: \mathcal{F} \rightarrow [0,1]$

① $\mathbb{P}(\Omega) = 1$

② $E_m \cap E_n = \phi \quad m \neq n \quad E_m \in \mathcal{F} \quad \forall n \geq 1$

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$$

Given $(\Omega, \mathcal{F}, \mathbb{P})$ - understand basic properties

Facts: [Proof - Exercise]

Ex. (i) $\mathbb{P}\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mathbb{P}(E_i) \quad E_1, \dots, E_n \in \mathcal{F}$
 $E_m \cap E_n = \phi \quad \forall m \neq n$

Ex (ii) $A \in \mathcal{F}, B \in \mathcal{F} \quad A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

(iii) $A_n \in \mathcal{A}_{n+1}, A_n \in \mathcal{F}, \forall n \geq 1$

$$A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$$

$$\mathbb{P}(A) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$$

[True:
Positive
measure]

[Ex: $\exists D_i \quad D_i \in \mathcal{F} \quad D_i \cap D_j = \phi \quad \forall i \neq j$
and $A_n = \bigcup_{m=1}^n D_m$]

• Use countable additivity there after

$$(iv) B_1 \supseteq B_2 \supseteq \dots \quad B_n \in \mathcal{F}$$

(not true for all positive measures)

$$\mathbb{P} \left(\bigcap_{n=1}^{\infty} B_n \right) = \lim_{n \rightarrow \infty} \mathbb{P}(B_n)$$

Counter Example:
 $(\mathcal{B}, \mathcal{B}_R, \lambda)$
 $B_n = [n, \infty)$

$$(v) C_n \in \mathcal{F} \quad \forall n \geq 1$$

$$\cdot \mathbb{P} \left(\bigcup_{m=1}^n C_m \right) \leq \sum_{k=1}^n \mathbb{P}(C_k)$$

$$\cdot \mathbb{P} \left(\bigcup_{m=1}^{\infty} C_m \right) \leq \sum_{n=1}^{\infty} \mathbb{P}(C_n)$$

↑
 $[0, \infty) \cup \{\infty\}$

$$(vi) A, B \in \mathcal{F}$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

- general statement available for $\{A_k\}_{k=i}^j \in \mathcal{F}$

$$(o) \mathbb{P}(\emptyset) = 0, \quad \mathbb{P}(A) = 1 - \mathbb{P}(A^c)$$

Existence of Uniform $[0,1]$

Existence & "uniqueness" $\left\{ \begin{array}{l} \Omega = [0,1] \quad \mathcal{B} = \text{Borel } \sigma\text{-algebra on } [0,1] \\ \mathbb{P}: \mathcal{B} \rightarrow [0,1] \quad \text{s.t.} \\ \cdot \mathbb{P}(a,b) = b-a \quad 0 \leq a \leq b \leq 1 \\ \cdot E_m \cap E_n = \emptyset \quad E_n \in [0,1] \text{ then } \mathbb{P} \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n) \end{array} \right.$

Proof (Ex.): Outer measures from algebras
 [statement] Caratheodory Extension Theorem

Borel σ -algebra on $[0,1]$

Ex:- $\mathcal{B} :=$ Smallest σ -algebra that contains all $\{(a,b) \mid 0 \leq a \leq b \leq 1\}$
 $= \sigma \{(a,b) \mid 0 \leq a \leq b \leq 1\}$

Regularity :- $E \in \mathcal{B} \equiv$ Borel σ -algebra on $[0,1]$
 $\mathbb{P} \equiv$ Uniform $[0,1]$

- $\mathbb{P}(E) = \sup \{ \mathbb{P}(K) : K \subseteq E \text{ } K\text{-compact} \}$
- $\mathbb{P}(E) = \inf \{ \mathbb{P}(U) : E \subseteq U \text{ } U\text{-open} \}$
- Given $\epsilon > 0 \exists$ K -compact U -open
 $K \subseteq E \subseteq U : \mathbb{P}(U \setminus K) < \epsilon$

Ex: Proposition 1.7.2 in [AS]

Bernoulli Trials

"Toss a fair coin finitely many times"

$$\Omega_n = \{\omega_i : \omega_i \in \{0,1\} \text{ } 1 \leq i \leq n\}$$

$$\mathcal{F}_n = \mathcal{P}(\Omega_n)$$

$$\mathbb{P} \equiv \mathbb{P}_n : \mathcal{F}_n \rightarrow [0,1]$$

$$\mathbb{P}(A) = \frac{|A|}{2^n} \text{ } \& \text{ } \text{ctly additive}$$

"Toss a fair coin infinitely many times"

$$\Omega = \{0,1\}^{\mathbb{N}}$$

$$\mathcal{F} \equiv ?$$

$$\mathbb{P} \equiv ?$$

$n \rightarrow \infty$