Probability Theory [Second Course in Measure Theoretic Probability]

I
- Probability Theory [as seen in undergraduate classes]
- Measure Theory – Review

II Build on the machine established in I.

Basics :-
- H.w. due every week [Moodle]
- Quiz every week (Thursday) [Question from the H.w.]
- Midterm / Final (Straightforward)

Website : Www.isibang.ac.in/~athreya/Teaching/c12

Notes, Scoring, References, Text, H.W, Syllabi for the week

+ +

Question / Direction of study :-

Q – Need for Measure Theory ?

Measure Theory :-

Sample Space

$\Omega :=$ any non empty set

[Probability :- Set of all possible outcomes of an Experiment

$\mathcal{F} \subseteq \mathcal{P}(\Omega) :=$ Power set of Ω.

[σ-algebra / σ-field]

(i) $\phi, \Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_n \in \mathcal{F} \;\forall\; n \geq 1$ then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$$

[∞ $\longrightarrow$ by $m \geq 1$ algebra or field]

[Probability :- Event (subcollection of outcomes)]

$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$

① $P(\Omega) = 1$

② $E_n \in \mathcal{F}$, $E_m \cap E_\ell = \phi$ for $m \neq \ell$

$$\mathbb{P}\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$$

[Probability :- function on sets satisfying Axioms ① and ②]

Examples:-

Experiment 1:- Toss a (fair) coin and let

$$X = \begin{cases} 1 & \text{if Head appears} \\ 0 & \text{if Tail appears} \end{cases}$$

$\Omega = \{H, T\}$

$\mathcal{F} = \mathcal{P}(\Omega) := \{\phi, \Omega, \{H\}, \{T\}\}$

[Model] $\mathbb{P} : \mathcal{F} \to [0,1]$ (Probability)

fair

$\mathbb{P}(\{H\}) = \frac{1}{2} = \mathbb{P}(\{T\})$

$X : \Omega \to \{0, 1\} = T$

$\mathcal{F}' = \{\phi, T, \{0\}, \{1\}\}$

$\mathbb{Q}(\{1\}) = \mathbb{P}(X = 1) = \frac{1}{2}$

$\mathbb{Q}(\{0\}) = \mathbb{P}(X = 0) = \frac{1}{2}$

$$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X} (T, \mathcal{F}', \mathbb{Q})$$

random variable

Experiment 2: Let $n \geq 1$ be given.

Toss a fair coin say n-times.

Ex: [Model]

$$\Omega = \{H, T\}^n$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$\mathbb{P} : \mathcal{F} \to [0,1]$$

... ?

X_n = Counts the # of Heads in n-tosses

$$T = \{0, 1, 2, \ldots, n\}$$

$\mathcal{F}' = ?$ $\qquad \mathbb{Q} : \mathcal{F}' \to [0,1]$

Show: $\mathbb{Q}_n(\{k\}) = {}^nC_k \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2}\right)^k.$

$k = 0, 1, 2, \ldots, n$

Binomial distribution

Theorem - P :- Coin with (biased) probability of heads $= p/n$

$$\mathbb{Q}_n(\{k\}) = {}^nC_k \, p^k (1-p)^{n-k} \xrightarrow[n \to \infty]{} \frac{e^{-p} p^k}{k!}$$

(Toss a coin n times)

bias $- p/n$

$\forall k = 0, 1, 2, \ldots$

Poisson (p) Distribution :- $\Omega = \mathbb{N} \cup \{0\}$, $\mathcal{F} = \mathcal{P}(\mathbb{N} \cup \{0\})$

$$\mathbb{Q}_p : \mathcal{F} \longrightarrow [0,1]$$

by $\quad \mathbb{Q}_p(\{k\}) := \dfrac{e^{-p} p^k}{k!} \quad \forall k \in \Omega$

$$\left[A \in \mathcal{F} \quad \mathbb{Q}_p(A) := \sum_{k \in A} \mathbb{Q}_p(\{k\}) \right]$$

$\mathbb{P}(X = k) \equiv X$ - Poisson Random variable

Theorem (C.L.T.) :- Toss a coin n times (fair)

$X_n = \#$ of Heads in n-tosses)

$$\mathbb{P}\left(\frac{a - \frac{n}{2}}{\sqrt{\frac{n}{4}}} < X_n \leq \frac{b - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \right) \xrightarrow[n \to \infty]{} \int_a^b \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx$$

(Normal distribution) $\Omega = \mathbb{R}$

$\mathcal{F} = \mathcal{B}(\mathbb{R})$ — Borel σ-algebra on $\mathbb{R}$

$\mathbb{P} : \mathcal{F} \rightarrow [0,1]$ - [Probability]

- Exists $\mathbb{P}$
- Unique!

$$\mathbb{P}((a,b]) = \int_a^b \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx$$

$\mathbb{Q}(X \in (a,b])$ $\quad X$ - Normal random variable

Undergraduate Probability - classification

[Discrete Random variables]

X: Bernoulli (p) Binomial(p) Poisson (λ)

[Continuous Random variables]

X: Normal $(0,1)$, Uniform $[0,1]$

Define: (Neither discrete nor continuous)

Toss a coin (fair)

if heads then $W \overset{d}{=}$ Poisson (λ)

if Tails then $W \overset{d}{=}$ Normal $(0,1)$

Ex: $(\Omega, \mathcal{F}, P)$ that will specify distribution of W.

First Motivation: - Set up Abstract measure theory.

Example : $X :=$ uniform chosen number in $[0,1)$

$\Omega = [0,1)$ $\mathcal{F} = ?$ $\mathbb{P} : \mathcal{F} \to [0,1]$

- $\mathbb{P}([a,b]) = b-a$ $0 \le a \le b \le 1$ — ①

(Proportional to length of interval)

[finite additivity]

- A, B to be two disjoint interval

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ — ②

[Countable additivity]

- $\{E_n\}_{n \ge 1}$ $E_m \cap E_\ell = \phi$ then

$$\mathbb{P}\Big(\underbrace{\bigcup_{m=1}^{\infty} E_m}_{\text{well defined Event}}\Big) = \underbrace{\sum_{m=1}^{\infty} \mathbb{P}(E_m)}_{\text{Series Converges.}} \quad \text{— ③}$$

Questions :-

(i) $\mathbb{P}$ — have uncountable additivity ?

(ii) Can there exists a $\mathbb{P}$ on $\mathcal{P}(\Omega)$

s.t. $\to$ ①, ②

$\to$ ①, ②, ③

Recall:

Q - need for Measure Theory

- why are each of the properties required ?

· Probability Distributions

Bernoulli (p)

Binomial (n, p)

Poisson (p)

Normal $(0,1)$

Need for Abstract Measure Theory

mixture of Random variables

Ω, $\mathcal{P}(\Omega) \equiv$ "too large"

Need for $\mathcal{F}$

Need for $\mathbb{P}$

4-8-2022 :-

Ω - non-empty set

$\mathcal{F}$ - σ-algebra $\equiv \begin{cases} \Omega, \phi \in \mathcal{F} \\ \text{- closed under compliment} \\ \text{- closed under countable unions.} \end{cases}$

$\mathbb{P}: \mathcal{F} \longrightarrow [0,1]$ ① $\mathbb{P}(\Omega) = 1$

② $E_m \cap E_n = \phi \quad m \neq n \qquad E_m \in \mathcal{F} \quad \forall n \geq 1$

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$$

Given $(\Omega, \mathcal{F}, \mathbb{P})$ - understand basic properties

Facts: [Proof - Exercise]

Ex∘ (i) $\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i) \qquad E_1, \ldots, E_n \in \mathcal{F}$

$E_m \cap E_n = \phi \quad \forall m \neq n$

Ex (ii) $A \in \mathcal{F}, \quad B \in \mathcal{F} \quad A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

(iii) $A_n \subseteq A_{n+1}, \quad A_n \in \mathcal{F}, \quad \forall n \geq 1$

$A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

$\mathbb{P}(A) = \lim_{n \to \infty} \mathbb{P}(A_n)$

[True: Positive measure]

[Ex: ∘ $\exists D_i \quad D_i \in \mathcal{F} \quad D_i \cap D_j = \phi \quad \forall i \neq j$

and $A_n = \bigcup_{m=1}^{n} D_m$]

- use countable additivity there after

(iv) $B_1 \supseteq B_2 \supseteq \cdots \qquad B_n \in \mathcal{F}$

(not true for all positive measures)

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n)$$

[Counter Example: $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$

$B_n = [n, \infty)$]

(v) $C_n \in \mathcal{F} \quad \forall n \geq 1$

- $$\mathbb{P}\left(\bigcup_{m=1}^{n} C_m\right) \leq \sum_{k=1}^{n} \mathbb{P}(C_k)$$

- $$\mathbb{P}\left(\bigcup_{m=1}^{\infty} C_n\right) \leq \sum_{m=1}^{\infty} \mathbb{P}(C_m)$$

$\uparrow$

$[0,\infty) \cup \{\infty\}$

(vi) $A, B \in \mathcal{F}$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

- General statement available for $\{A_k\}_{k=1}^{n} \in \mathcal{F}$

(0) $\mathbb{P}(\phi) = 0, \qquad \mathbb{P}(A) = 1 - \mathbb{P}(A^c)$

Existence of Uniform [0,1)

Existence & "uniqueness"

$\Omega = [0,1] \qquad \mathcal{B} =$ Borel σ-algebra on $[0,1]$

$\mathbb{P} : \mathcal{B} \to [0,1]$ s.t.

- $\mathbb{P}((a,b]) = b - a \qquad 0 \leq a \leq b \leq 1$

- $E_m \cap E_n = \phi \quad E_n \in [0,1]$ then $\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$

Proof (Ex.):
- Outer measures from algebras
- [statement] Caratheodory Extension Theorem
- Borel σ-algebra on [0,1]

$\mathcal{B} :=$ Smallest σ-algebra that Contains all $\{(a,b] \mid 0 \le a \le b \le 1\}$

Ex:- $= \sigma\{(a,b] \mid 0 \le a \le b \le 1\}$

Regularity :- $E \in \mathcal{B} \equiv$ Borel σ-algebra on $[0,1)$

$\mathbb{P} \equiv$ Uniform $[0,1)$

- $\mathbb{P}(E) = \sup\{\mathbb{P}(K) : K \subseteq E \quad K\text{-compact}\}$
- $\mathbb{P}(E) = \inf\{\mathbb{P}(U) : E \subseteq U \quad U\text{-open}\}$
- Given $\varepsilon > 0$ $\exists$ K-compact & U-open

$K \subseteq E \subseteq U \quad : \mathbb{P}(U \setminus K) < \varepsilon$

Ex: Proposition 1.7.2 in [AS]

Bernoulli Trials

"Toss a fair coin finitely many times"

$\Omega_n = \{(w_i) : w_i \in \{0,1\} \; 1 \le i \le n\}$

$\mathcal{F}_n = \mathcal{P}(\Omega_n)$

$\mathbb{P} \equiv \mathbb{P}_n : \mathcal{F}_n \to [0,1]$

$\mathbb{P}(A) = \frac{|A|}{2^n}$ & c'tbly additive

$\boxed{n \to \infty}$

"Toss a fair coin infinitely many times"

$\Omega = \{0,1\}^{\mathbb{N}}$

$\mathcal{F} = ?$

$\mathbb{P} = ?$