Due Date: October 19th 2022, 10pm
Problems Due: 1,3,4

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space.

1. Suppose $\{X_n\}_{n\geq 1}$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$such that $X_n \xrightarrow{p} X$ and $X_n \xrightarrow{p} Y$ as $n \to \infty$ then show that $\mathbb{P}(X = Y) = 1$.

2. Suppose $\{X_n\}_{n\geq 1}$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Then show that $X_n \xrightarrow{a.e.} X$ if and only if for all $\epsilon > 0$

$$\mathbb{P}(\mid X_n - X \mid > \epsilon \text{ infinitely often}) = 0.$$

3. Suppose $\{X_n\}_{n\geq 1}$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that for all $\epsilon > 0$

$$\mathbb{P}(\sup_{m\geq n} \mid X_m - X_n \mid > \epsilon) \to 0 \text{ as } n \to \infty$$

$$\Rightarrow$$

$$\sup_{m\geq n} \mathbb{P}(\mid X_m - X_n \mid > \epsilon) \to 0 \text{ as } n \to \infty.$$

   Is the converse true ?

4. Let Z_n be i.i.d random variables on $(\Omega, \mathcal{F}, \mathbb{P})$such that

$$\mathbb{P}(Z_n = 1) = \frac{1}{2} = 1 - \mathbb{P}(Z_n = 0).$$

   Define $X_n = \frac{Z_n}{n^\theta}$ for $0 < \theta$. Decide whether the series with partial sums $S_n = \sum_{j=1}^n X_n$ converges almost surely or not ?

5. Suppose $\{X_n\}_{n\geq 1}, X$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}[X_n^2] < \infty, \forall n \geq 1, \quad \text{and} \quad \mathbb{E}[(X_n - X)^2] \to 0 \text{ as } n \to \infty.$$

   Show that $\mathbb{E}[X_n^2] \to \mathbb{E}[X^2]$.

6. Suppose $\{X_n\}_{n\geq 1}, X$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$such that $X_n \xrightarrow{a.e.} X$ as $n \to \infty$. Show that there is a $K > 0$ large enough that

$$\mathbb{P}(\sup_{n\geq 1} \mid X_n \mid < K) > 0.$$