1. Exercise 5.2.18 in [AS08].

   Solution: $\{Y_n\}_{n\geq 1}$ are i.i.d random variables uniformly distributed over $(0,\theta)$.

   Let, $X_n = \max\{Y_1, Y_2, \ldots, Y_n\}$. Then given $\epsilon > 0$ we have,

$$\begin{aligned}\mathbb{P}(|X_n - \theta|| \geq \epsilon) &= \mathbb{P}(X_n \leq \theta - \epsilon \text{ or } X_n \geq \theta + \epsilon)\\ &= \mathbb{P}(X_n \leq \theta - \epsilon) \quad [\text{becasue } X_n \leq \theta \quad \forall n]\\ &= \mathbb{P}(Y_n \leq \theta - \epsilon, 1 \leq k \leq n)\\ &= \prod_{k=1}^{n} \mathbb{P}(Y_k \leq \theta - \epsilon) \quad [\text{By independence of } Y_k\text{'s}]\\ &= \Big(\frac{\theta - \epsilon}{\theta}\Big)^n \quad [\text{If } \epsilon \leq \theta. \text{ And if } \epsilon > \theta \text{ this is 0 anyway}]\end{aligned}$$

   But, $0 \leq \frac{\theta-\epsilon}{\theta} < 1$ if $\epsilon < \theta$.

   Thus,

$$\lim_{n\to\infty} \Big(\frac{\theta - \epsilon}{\theta}\Big) = 0$$

   Then,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - \theta|| \geq \epsilon) = 0.$$

   Thus,

$$X_n \xrightarrow{p} \theta \text{ as } n \to \infty.$$

2. Exercise 5.2.21 in [AS08].

   Solution: Let $F(x) = \mathbb{P}(Y_n \leq x)$ so that F is the common cumulative distribution function for all the Y_n's.

   Then,

$$\begin{aligned}\mathbb{P}(|X_n| \geq \epsilon) &= \mathbb{P}(|Y_n| \geq n\epsilon)\\ &= \mathbb{P}(Y_n \geq n\epsilon \text{ or } Y_n \leq -n\epsilon)\\ &= F(-n\epsilon) + 1 - F(n\epsilon)\end{aligned}$$

   Now, F satisfies $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

   Thus, given $\delta > 0$ choose $n_{\epsilon_0} \in \mathbb{N}$ so that $\forall n \geq n_0$ we have

$$F(-n\epsilon) < \frac{\delta}{2} \text{ and } 1 - F(n\epsilon) < \frac{\delta}{2}.$$

   Then, $\forall n \geq n_0$ we have, $\mathbb{P}(|X_n| \geq \epsilon) < \delta$.

   Thus, $\lim_{n\to\infty} \mathbb{P}(|X_n| \geq \epsilon) = 0$.

   Thus,

$$X_n \xrightarrow{p} 0.$$

   We now do the second part of the problem. As $X_n \xrightarrow{p} 0$ if $X_n \xrightarrow{a.e} Y$ then Y must be 0.

We can construct a random variable $Z : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{N}$ such that

$$\mathbb{P}(Z = i) = \frac{i}{i+1}.$$

Then, let Y_n be i.i.d copies of Z $\forall n \geq 1$. We have $X_n = \frac{Y_n}{n}$ $\forall n \geq 1$.

Now,

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq 1) = \sum_{n=1}^{\infty} [\sum_{k=n}^{\infty} \frac{1}{k(k+1)}] = \infty.$$

Thus, $\mathbb{P}(\lim_{n\to\infty} X_n = 0) \neq 1$ [By Borel Cantelli lemma]

Thus, X_n does not converge almost everywhere.

6. Exercise 5.5.11 in [Ros06].

   Solution: Let $\Omega = [0, 1]$ with uniform probability measure. Define $Y_n : \Omega \to \mathbb{R}$ as follows:

$$Y_{2^{k+1}-2+m} = (2^{k+1} - 2 + m) I_{[\frac{m-1}{2^{k+1}}, \frac{m}{2^{k+1}}]}$$

with $k \geq 0$ and $1 \leq m \leq 2^{k+1}$.

Now, given ϵ, if $n > 2^k$ we have,

$$\mathbb{P}(|\frac{Y_n}{n}| \geq \epsilon) \leq \frac{1}{2^{k+1}}.$$

Thus, given $\delta > 0$ choose k with $\frac{1}{2^{k+1}} < \delta$. Then, for $n > 2^k = n_0$ we have,

$$\mathbb{P}(|\frac{Y_n}{n}| \geq \epsilon) < \epsilon.$$

Thus,

$$\lim_{n\to\infty} \mathbb{P}(|\frac{Y_n}{n}| \geq \epsilon) = 0.$$

Thus,

$$\frac{Y_n}{n} \xrightarrow{p} 0.$$

But, if $\omega \in \Omega$, the sequence $\{\frac{Y_n(\omega)}{n}\}$ takes the value 1 infinitely often. Thus, $\frac{Y_n(\omega)}{n}$ can not converge to 0.

So $\{\omega \in \Omega | \lim_{n\to\infty} \frac{Y_n(\omega)}{n} = 0\}$ implies,

$$\mathbb{P}(\lim_{n\to\infty} \frac{Y_n}{n} = 0) = 0.$$

So, $\frac{Y_n}{n}$ does not converge to 0 in probability. But if $\omega \in \Omega$, then

$\frac{Y_n(\omega)}{n^2} \leq \frac{1}{n}$ and hence

$$\lim_{n\to\infty} \frac{Y_n(\omega)}{n^2} = 0.$$

Thus,

$$\begin{aligned}
&\{\omega \in \Omega | \lim_{n\to\infty} \frac{Y_n(\omega)}{n^2} = 0\} = \Omega \\
\Rightarrow &\mathbb{P}(\lim_{n\to\infty} \frac{Y_n}{n^2} = 0) = 1 \\
\Rightarrow &\frac{Y_n}{n^2} \overset{a.s}{\to} 0 \text{ as } n \to \infty.
\end{aligned}$$

8. Exercise 5.5.15 in [Ros06].

   Solution of (a): Let X_n's be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $E_n = X_n^{-1}\{3^n\}, E_n^{'} = X_n^{-1}\{-3^n\}$ then $X_n = 3^n I_{E_n} - 3^n 1_{E_n'}$.
   Thus,

$$\mathbb{E}[X_n] = 3^n \mathbb{P}(E_n) - 3^n \mathbb{P}(E_n^{'}) = 0.$$

   Solution of (b):

$$\begin{aligned}
|S_n| &= |X_1 + X_2 + \ldots + X_n| \\
&\geq |X_n| - |X_{n-1}| - |X_{n-2}| - \ldots - |X_1| \\
&\geq 3^n - \sum_{k=1}^{n-1} 3^k \\
&= 3^n - \frac{3^n - 3}{2} \\
&= \frac{3^n + 3}{2}
\end{aligned}$$

   Thus,

$$\mathbb{P}(|S_n| \geq \frac{3^n + 3}{2}) = 1.$$

   But note that,

$$\begin{aligned}
\mathbb{P}(|S_n| = \frac{3^n + 3}{2}) &\geq \mathbb{P}(X_n = 3^n, X_{n-1} = -3^{n-1}, X_{n-2} = -3^{n-2}, \ldots, X_2 = -3^2, X_1 = -3) \\
&= \mathbb{P}(X_n = 3^n)\mathbb{P}(X_{n-1} = -3^{n-1}) \ldots \mathbb{P}X_2 = -3^2)\mathbb{P}(X_1 = -3) \text{[By independence]} \\
&= \frac{1}{2^n} \\
&> 0
\end{aligned}$$

   Thus if $r > \frac{3^n+3}{2}$ we have,

$$\mathbb{P}(|S_n| \geq r) = 1 - \mathbb{P}(|S_n| < r) < 1.$$

   Thus, $R_n = \frac{3^n+3}{2}$.

   Solution of (c): We have, $R_n = \frac{3^n+3}{2}$. Then,

$$\begin{aligned}
\lim_{n\to\infty} \frac{R_n}{n} &= \lim_{n\to\infty} \frac{3^n + 3}{2n} \\
&= \lim_{n\to\infty} \frac{3^n}{2n} \\
&= \infty
\end{aligned}$$

Solution of (d): Given $\epsilon > 0$, we choose n_0 so that $\frac{R_n}{n} > \epsilon \; \forall n \geq n_0$. Then, $\forall n \geq n_0$ we have,

$$\mathbb{P}(\frac{|S_n|}{n} \geq \epsilon) \leq \mathbb{P}(\frac{|S_n|}{n} \geq \frac{R_n}{n}) = 1$$

This implies,

$$\lim_{n\to\infty} \mathbb{P}(\frac{|S_n|}{n} \geq \epsilon) = 1 \neq 0 \quad \forall \epsilon > 0.$$

Solution of (e): The random variables $\{X_n\}_{n\geq 1}$ are not i.i.d and do not have a uniform bound for the variances or moments. So this does not contradict the laws of large numbers.

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