1. Exercise 4.5.13 in [Ros06]

**Solution of (a):** We first define $X^+$ and $X^-$. Then, we take $X = X^+ - X^-$.

Define $X^+ : [0,1] \to \mathbb{R}$ as:

$$X^+(\omega) = \begin{cases} 2 & \text{if } \frac{1}{3} < \omega \leq \frac{1}{2} \\ 3 & \text{if } \frac{1}{4} < \omega \leq \frac{1}{3} \\ \text{and generally } k & \text{if } \frac{1}{k+1} < \omega \leq \frac{1}{k} \\ 0 & \text{otherwise} \end{cases}$$

Define $X^- : [0,1] \to \mathbb{R}$ as:

$$X^-(\omega) = \begin{cases} 1 & \text{if } \frac{1}{2} < \omega \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, $X$ is

$$X(\omega) = \begin{cases} -1 & \text{if } \frac{1}{2} < \omega \leq 1 \\ 2 & \text{if } \frac{1}{3} < \omega \leq \frac{1}{2} \\ 3 & \text{if } \frac{1}{4} < \omega \leq \frac{1}{3} \\ \text{and generally } k & \text{if } \frac{1}{k+1} < \omega \leq \frac{1}{k} \\ 0 & \text{if } \omega = 0 \end{cases}$$

Define, $X_m^+ = X^+ I_{\{\frac{1}{m} \leq \omega \leq 1\}}$. Then, $X_m^+$ is non-negative simple function such that $X_m^+ \uparrow X^+$. Now, by Monotone convergence theorem , $\mathbb{E}[X_m^+] \uparrow \mathbb{E}[X^+]$.

Then,

$$\mathbb{E}(X_m^+) = \sum_{k=2}^{m} \frac{k}{k(k+1)} \text{ and } \mathbb{E}(X^-) = \frac{1}{2} \implies 0 < \mathbb{E}(X^-) < \infty.$$

Then,

$$\mathbb{E}(X^+) = \sum_{k=2}^{\infty} \frac{k}{k(k+1)} = \infty \text{ and } \mathbb{E}(X^-) = \frac{1}{2} \implies 0 < \mathbb{E}(X^-) < \infty.$$

**Solution of (b):** Let us use $X, X^+$ and $X^-$ from part **(a)**. We define, $Y = -X$. Then, we have

$$Y^- = X^+ \text{ and } Y^+ = X^-.$$

This implies,

$$\mathbb{E}(Y^-) = \infty \text{ and } 0 < \mathbb{E}(Y^+) < \infty.$$

**Solution of (c):** We first define $X^+$ and $X^-$. Then, we take $X = X^+ - X^-$.

Define $X^+ : [0,1] \to \mathbb{R}$ as:

$$X^+(\omega) = \begin{cases} k & \text{if } \frac{1}{k+1} < \omega \leq \frac{1}{k} \text{ and } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Define $X^- : [0,1] \to \mathbb{R}$ as:

$$X^-(\omega) = \begin{cases} k & \text{if } \frac{1}{k+1} < \omega \leq \frac{1}{k} \text{ and } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Then, $X$ is

$$X(\omega) = \begin{cases} k & \text{if } \frac{1}{k+1} < \omega \leq \frac{1}{k} \text{ and } k \text{ is odd} \\ -k & \text{if } \frac{1}{k+1} < \omega \leq \frac{1}{k} \text{ and } k \text{ is even} \end{cases}$$

Define, $X_m^+ = X^+ I_{\{\frac{1}{m} \leq \omega \leq 1\}}$ $\forall m \geq 1$. Then, $X_{2m}^+$ is non-negative simple function such that $X_{2m}^+ \uparrow X^+$. Similarly, $X_{2m+1}^-$ is non-negative simple function such that $X_{2m+1}^- \uparrow X^-$.
Then

$$\mathbb{E}(X_{2m}^+) = \sum_{n=1}^{m} \frac{2n-1}{(2n-1)2n} \text{ and } \mathbb{E}(X_{2m+1}^-) = \sum_{n=1}^{m} \frac{2n}{2n(2n+1)}.$$

Now, by Monotone convergence theorem , $\mathbb{E}[X_m^+] \uparrow \mathbb{E}[X^+]$ and $\mathbb{E}[X_m^-] \uparrow \mathbb{E}[X^-]$.
Then

$$\mathbb{E}(X^+) = \sum_{n=1}^{\infty} \frac{2n-1}{(2n-1)2n} = \infty \text{ and } \mathbb{E}(X^-) = \sum_{n=1}^{\infty} \frac{2n}{2n(2n+1)} = \infty.$$

**Solution of (d):** Consider the Pareto distribution with density function

$$f_X(x) = \frac{\alpha x_m^\alpha}{x^{1+\alpha}} \text{ with } \alpha \in (1,2], x \in [x_m, \infty).$$

Then

$$\begin{aligned} \mathbb{E}[X] &= \int_{x_m}^{\infty} \frac{\alpha x_m^\alpha}{x^\alpha} dx \\ &= \frac{\alpha x_m^\alpha}{(1-\alpha)x^{\alpha-1}}\Big|_{\infty}^{x_m} \\ &= \frac{\alpha x_m}{\alpha - 1} \end{aligned}$$

But, $\mathbb{E}[X^2] = \infty$.

2. (Tschebychev Inequality)

   (a) Find a random variable $X$ with $\text{Range}(X) = \{-1, 0, 1\}$ such that

   $$P(\mid X - \mu \mid \geq 2\sigma) = \frac{1}{4},$$

   with $\mu = E[X]$ and $\sigma^2 = \text{Var}[X]$.

(b) Construct another random variable $Y$ (different from $X$) with Range $(Y) = \{y_1, y_2, y_3\}$, $E[Y] = \mu$ and

$$P(\mid Y - \mu \mid > 2\sigma) > P(\mid X - \mu \mid > 2\sigma),$$

so as to get

$$P(\mid Y - \mu \mid > 2\sigma) > \frac{1}{4}$$

Decide whether Tschebychev Inequality is violated ?

**Solution of (a):** $X : [0,1] \to \mathbb{R}$ where $[0,1]$ has uniform measure.

Let

$$X(\omega) = \begin{cases} -1 & \text{if } \dfrac{1}{k+1} < \omega \leq \dfrac{1}{k} \text{ and } k \text{ is odd} \\ 0 & \text{if } \dfrac{1}{k+1} < \omega \leq \dfrac{1}{k} \text{ and } k \text{ is even} \\ 1 & \text{if } \dfrac{3}{4} < \omega \leq 1 \end{cases}$$

Then, $\mathbb{E}[X] = -\frac{1}{2} = \mu$ and $\sigma = Var[X] = 1 - \frac{1}{4} = \frac{3}{4}$ and $\mathbb{P}(|X + \frac{1}{2}| \geq \frac{3}{2}) = \mathbb{P}(X = 1) = \frac{1}{4}$.

**Solution of (b):** Let $Y : [0,1] \to \mathbb{R}$

$$Y(\omega) = \begin{cases} -2 & \text{if } 0 \leq \omega < \dfrac{5}{8} \\ 0 & \text{if } \omega = \dfrac{5}{8} \\ 2 & \text{if } \dfrac{5}{8} < \omega \leq 1 \end{cases}$$

Then, $E[Y] = -2 \cdot \frac{5}{8} + 2 \cdot \frac{3}{8} = -\frac{1}{2} = \mu$ and $\mathbb{P}(|Y + \frac{1}{2}| \geq 2 \cdot \frac{3}{4}) = \mathbb{P}(Y = 2) = \frac{3}{8} > \frac{1}{4}$.

Since $Var(Y) \neq \sigma$, Tschebychev inequality in not violated.

5. Exercise 3.1.12 [AS08]

**Solution:** $X : \Omega \to \mathbb{N}$ is a random variable. Then,

$$E[X] = \sum_{n=1}^{\infty} n\mathbb{P}(X = n) = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \mathbb{P}(X = k) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$$

6. Exercise 4.3.4 [Ros06]

**Solution of (a):** If $X$ and $Y$ are independent and $f$ and $g$ are measurable functions then $f(X)$ and $f(Y)$ are independent random variables as:

$$\begin{aligned} \mathbb{P}(f(X) \leq x, g(Y) \leq y) &= \mathbb{P}(X \in f^{-1}(-\infty, x], Y \in g^{-1}(-\infty, y]) \\ &= \mathbb{P}(X \in f^{-1}(-\infty, x])\mathbb{P}(Y \in g^{-1}(-\infty, y]) \text{ (X and Y are independent)} \\ &= \mathbb{P}(f(X) \leq x)\mathbb{P}(g(Y) \leq y) \end{aligned}$$

Now if $f(x) = max\{0, x\}$ and $g(x) = max\{0, -x\}$ then $f$ and $g$ are measurable. Hence $X^+ = f(X)$, $Y^+ = f(Y), X^- = g(X)$, and $Y^- = g(Y)$.

Thus, $X^+, Y^+, X^-, Y^-$ are mutually independent to each other.

**Solution of (b):**

$$\begin{aligned} Z^+ - Z^- &= (X^+ - X^-)(Y^+ - Y^-) \\ &= X^+Y^+ + X^-Y^- - (X^+Y^- + X^-Y^+) \end{aligned}$$

Now, $X^+Y^+ + X^-Y^- \geq 0$ and $X^+Y^- + X^-Y^+ \geq 0$. Thus,

$$\begin{aligned} Z^+ &= X^+Y^+ + X^-Y^- \\ Z^- &= X^+Y^- + X^-Y^+ \end{aligned}$$

**Solution of (c):**

$$\begin{aligned} \mathbb{E}(XY) &= \mathbb{E}((X^+ - X^-)(Y^+ - Y^-)) \\ &= \mathbb{E}(X^+Y^+ + X^-Y^- - X^-Y^+ - X^+Y^-) \\ &= \mathbb{E}(X^+Y^+) + \mathbb{E}(X^-Y^-) - (\mathbb{E}(X^-Y^+) + \mathbb{E}(X^+Y^-)) \\ &= \mathbb{E}(X^+)\mathbb{E}(Y^+) + \mathbb{E}(X^-)\mathbb{E}(Y^-) - \mathbb{E}(X^-)\mathbb{E}(Y^+) - \mathbb{E}(X^+)\mathbb{E}(Y^-)[\text{ by part (a)}] \\ &= \mathbb{E}(X^+ - X^-)\mathbb{E}(Y^+ - Y^-) \\ &= \mathbb{E}(X)\mathbb{E}(Y) \end{aligned}$$

7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space.

   a. Suppose $X, Y$ are simple non-negative random variables with $X \leq Y$ then show that $E[X] \leq E[Y]$.
   **Solution of (a):** Suppose
   $$X = \sum_{i=1}^{n} a_i I_{E_i}$$
   where $E_i \cap E_j = \varnothing$ and $Y = \sum_{j=1}^{n} b_j I_{E_j}$. $[a_i \geq 0$ and $b_j \geq 0]$ and $X \leq Y \implies a_i \leq b_i \quad \forall i = 1, \ldots, n$.
   We may assume $a_i$ and $b_i$ are distinct.
   Then, $\mathbb{E}[X] = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i = \mathbb{E}[Y]$.

   b. Suppose $X, Y$ are non-negative random variables with $X \leq Y$ then show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
   **Solution of (b):** Suppose $\phi_n(x) = \min\{n, 2^{-n}\lfloor 2^n x \rfloor\}$ where $n \geq 1$.
   Now since $X \leq Y \implies \phi_n(X) \leq \phi_n(Y)\ \forall n \geq 1$.[Becasue $\phi_n(x) \geq 0$ and $X \geq 0$]
   Now, $\phi_n(X) \uparrow X$ as $n \to \infty$. By monotone convergence theorem,
   $$\mathbb{E}[\phi_n(X)] \uparrow \mathbb{E}[X] \text{ as } n \to \infty$$
   and similarly
   $$\mathbb{E}[\phi_n(y)] \uparrow \mathbb{E}[Y] \text{ as } n \to \infty.$$
   But, by (a),
   $$\mathbb{E}[\phi_n(X)] \leq \mathbb{E}[\phi_n(Y)] \quad \forall n \geq 1.$$
   Thus,
   $$\mathbb{E}[X] \leq \mathbb{E}[Y].$$

c. Suppose $X, Y$ are random variables with $X \leq Y$ then show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$, provided both exists.

**Solution of (b):**

$$X^+(\omega) = \max\{0, X(\omega)\}$$
$$X^-(\omega) = \max\{0, -X(\omega)\}$$

Then
$$X \leq Y \implies X^+ \leq Y^+ \text{ and } X^- \geq Y^-.$$

Then, from part (b) we have,

$$\mathbb{E}[\phi_n(X^+)] \leq \mathbb{E}[\phi_n(Y^+)] \text{ and } \mathbb{E}[\phi_n(X^-)] \geq \mathbb{E}[\phi_n(Y^-)] \quad \forall n \geq 1.$$

Thus,

$$\mathbb{E}[\phi_n(X^+) - \phi_n(X^-)] \leq \mathbb{E}[\phi_n(Y^+) - \phi_n(Y^-)] \quad \forall n \geq 1$$

Then,

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X^+] - \mathbb{E}[X^-] \\
&= \lim_{n\to\infty} \mathbb{E}[\phi_n(X^+)] - \lim_{n\to\infty} \mathbb{E}[\phi_n(X^-)] \text{ [By monotone convergence]} \\
&\leq \lim_{n\to\infty} \mathbb{E}[\phi_n(Y^+)] - \lim_{n\to\infty} \mathbb{E}[\phi_n(Y^-)] \\
&= \mathbb{E}[Y^+] - \mathbb{E}[Y^-] \\
&= \mathbb{E}[Y]
\end{aligned}$$

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