3(a). Let $\{X_n\}_{n\geq 1}$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X_n \sim Normal(0,1)$ then show that

$$\mathbb{P}(\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2\log(n)}} = 1) = 1.$$

Solution : We first derive an inequality regarding tail probabilities of Gussian random variables

$$\begin{aligned}
\int_x^\infty e^{-\frac{t^2}{2}} dt &= \int_x^\infty \frac{1}{t} t e^{-\frac{t^2}{2}} dt \text{ if } x \geq 1 \\
&= -\frac{e^{\frac{t^2}{2}}}{t}|_x^\infty - \int_x^\infty \frac{e^{-\frac{t^2}{2}}}{t^2} dt \\
&= \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^\infty \frac{1}{t^3} t e^{-\frac{t^2}{2}} dt \\
&= \frac{e^{-\frac{x^2}{2}}}{x} + \frac{e^{-\frac{t^2}{2}}}{t^3}|_x^\infty + \int_x^\infty \frac{3e^{-\frac{t^2}{2}}}{t^4} dt
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_x^\infty e^{\frac{t^2}{2}} dt &\geq e^{-\frac{x^2}{2}} (\frac{1}{x} - \frac{1}{x^3}) \\
&\geq e^{-\frac{x^2}{2}} (\frac{1}{x} - \frac{1}{4x}) \\
&\geq \frac{3}{4x} e^{-\frac{x^2}{2}} \text{ if } x \geq 2
\end{aligned}$$

Again,

$$\begin{aligned}
\int_x^\infty e^{-\frac{t^2}{2}} dt &\leq \int_x^\infty \frac{t}{x} e^{-\frac{t^2}{2}} \text{ if } x \geq 1 \\
&= \frac{1}{x} e^{-\frac{x^2}{2}}
\end{aligned}$$

Thus, $\frac{3}{4x} e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{t^2}{2}} dt \leq \frac{1}{x} e^{-\frac{x^2}{2}}$ $\ldots(i)$

Now we show $\mathbb{P}(\lim_{n\to\infty} \sup \frac{x_n}{\sqrt{2\log n}} \leq 1) = 1$

Say $\epsilon > 0$ then

$$\begin{aligned}
\sum_{n\geq 2} \mathbb{P}(\frac{X_n}{\sqrt{2\log n}} \geq \sqrt{1+\epsilon}) &= \sum_{n\geq 2} \mathbb{P}(X_n \geq \sqrt{2(1+\epsilon)\log n} \\
&\leq \sum_{n\geq 2} \frac{1}{\sqrt{2\pi}\sqrt{2(1+\epsilon)\log n}}) \exp(-\frac{2(1+\epsilon)\log n}{2}) \text{ (by } (i)) \\
&\leq \sum_{n\geq 2} n^{-(1+\epsilon)} \\
&< \infty
\end{aligned}$$

Since $\{X_n\}$ are independent using Borel-Cantelli lemma we have

$$\mathbb{P}(\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2\log n}} > \sqrt{1+\epsilon}) = 0$$

Thus,

$$\mathbb{P}(\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2\log n}} \leq \sqrt{1+\epsilon}) = 1.$$

Suppose $A_m = \{\lim_{n\to\infty} \sup \frac{X_m}{\sqrt{2\log n}} \leq \sqrt{1+\frac{1}{m}}\}$ then $A_{m+1} \subset A_m$ and by the continuity of probability we have

$$\mathbb{P}(\cap_{m\geq 1} A_m) = \lim_{m\to\infty} \mathbb{P}(A_m) = 1$$

$$\Rightarrow \mathbb{P}(\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2\log n}} \leq 1) = 1$$

Again we show that $\mathbb{P}(\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2\log n}} \geq 1) = 1$.

Now,

$$\begin{aligned}
\sum_{n\geq 2} \mathbb{P}(\frac{X_n}{\sqrt{2\log n}} \geq 1) &= \sum_{n\geq 2} \mathbb{P}(X_n \geq \sqrt{2\log n}) \\
&\geq \sum_{n\geq 2} \frac{3}{4\sqrt{2\pi}\sqrt{2\log n}} \exp(-\log n) \\
&= \frac{3}{8\pi} \sum_{n\geq 2} \frac{1}{n\sqrt{\log n}} \\
&\geq \frac{3}{8\sqrt{\pi}} \int_2^\infty \frac{dx}{x\sqrt{\log x}} \\
&< \infty
\end{aligned}$$

Since X_n are independent using Borel-Cantelli lemma we have,

$$\mathbb{P}(\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2\log n}} \geq 1) = 1$$

Thus,

$$\mathbb{P}(\{\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2\log n}} \geq 1\} \cap \{\lim_{n\to\infty} \sup \frac{X_n}{\sqrt{2\log n}} \leq 1\}) = 1$$

because if $\mathbb{P}(A) = \mathbb{P}(B) = 1$ then $\mathbb{P}(A \cap B) = 1$.

4. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Suppose $\Omega = \{-1, 1\}^{\mathbb{N}}$ along with

$$\mathbb{P}(\{\omega \in \Omega | \pi_n(\omega) = (\tilde{\omega_1}, \tilde{\omega_2}, \ldots, \tilde{\omega_n})\}) = \frac{1}{2^n}$$

,

for any $(\tilde{\omega_1}, \tilde{\omega_2}, \ldots, \tilde{\omega_n}) \in \{-1, 1\}^n$. Show that

$$\mathbb{P}(\{\omega \in \Omega | \sum_{n=1}^{\infty} \omega_n a_n < \infty\}) \in \{0, 1\}.$$

Solution : Let $A_n \subset \Omega$ be the event $\{\omega_n = 1\}$. Considers distinct $n_1, n_2, \ldots, n_k \in \mathbb{N}$. Wlog assume $n_1 < n_2 < \ldots < n_k$.
$\mathbb{P}(A_{n_1} \cap A_{n_2} \cap \ldots \cap A_{n_k})$
$=\mathbb{P}(\{\omega \in \Omega | \pi_{n_k}(\omega) = (\omega_1, \ldots, \omega_{n_k})$ where $\omega_{n_1} = \omega_{n_2} = \ldots = \omega_{n_k} = 1\}) = \frac{1}{2^k}$
and $\mathbb{P}(A_n) = \frac{1}{2}\ \forall n$
Thus, $\mathbb{P}(A_{n_1} \cap A_{n_2} \cap \ldots \cap A_{n_k}) = \mathbb{P}(A_{n_1})\mathbb{P}(A_{n_2})\ldots\mathbb{P}(A_{n_k})$.
Thus, $\{A_n\}_{n\geq 1}$ are independent. If we show that

$$\{\omega \in \Omega | \sum_n \omega_n a_n < \infty\} \in \cap_{k\geq 1}\sigma(A_k, A_{k+1}, \ldots)$$

then by Kolmogorov's 0-1 law we will be done.
Now, $E = \{\omega \in \Omega | \sum_n \omega_n a_n < \infty\} = \cap_{k=1}^{\infty} \cup_{N\geq 1} \cap_{q>p\geq N} \{\omega \in \Omega || \sum_{n=p}^{q} \omega_n a_n| < \frac{1}{k}\}$
Observe, $\cap_{q>p\geq N}\{\omega \in \Omega || \sum_{n=p}^{q} \omega_n a_n| < \frac{1}{k}\} \in \sigma(A_{N'}, A_{N'+1}, \ldots)\ \forall N' \geq N$
Thus, $E \in \cap_{k\geq 1}\sigma(A_k, A_{k+1}, \ldots)\ \forall k$.
Thus, E is a tail event. So, $\mathbb{P}(E) = 0$.

6. (*Constructing independent and identically distributed (i.i.d.) sequence of Uniform random variables*) Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}_{[0,1]}$ the Borel-σ algebra on $[0,1]$, and $\mathbb{P}(dw)$ be the Lebesgue measure. Consider random variables $d_k : [0,1] \to \mathbb{R}$ with $d_k(\omega)$ being the k-th digit in the binary expansion of ω.

   (a) Show that $\{d_k\}_{k\geq 1}$ is an i.i.d. Bernoulli $(\frac{1}{2})$ sequence, i.e.

      i. Show that $\mathbb{P}(d_k = 0) = \mathbb{P}(d_k = 1) = \frac{1}{2}$ for all $k \geq 1$.
      ii. Show that $\{d_k\}_{k\geq 1}$ is an independent collection of random variables.

   Solution : If we fix the first $(k-1)$ digits of ω then the set of all ω with these $(k-1)$ first digits and $k-$th digit 0 forms an interval of length $\frac{1}{2^k}$ and as we vary the first $(k-1)-$digits we get 2^{k-1} disjoint intervals of equal length $\frac{1}{2^k}$.
   Thus, $\mathbb{P}(d_k = 0) = \frac{1}{2}$
   Similarly, we can show $\mathbb{P}(d_k = 1) = \frac{1}{2}$ again if $k_1 < k_2 < \ldots < k_n$.
   $\mathbb{P}(d_{k_1} = \omega_1, d_{k_2} = \omega_2, \ldots, d_{k_n} = \omega_n)$ [where $\omega_i \in \{0,1\}$]
   $= \mathbb{P}(\{\tilde{\omega} \in \Omega | \omega_{k_1} = \omega_1, \ldots, \omega_{k_n} = \omega_n\})$
   Again fixing the first k_n digits gives us an interval of length $\frac{1}{2^{k_n}}$ and the first k_n digits can vary over $2^{(k_n - n)}$ choices which give $2^{(k_n-n)}$ disjoint intervals of length $\frac{1}{2^{k_n}}$.
   Thus, $\mathbb{P}(d_{k_1} = \omega_1, d_{k_2} = \omega_2, \ldots, d_{k_n} = \omega_n) = \frac{2^{k_n - n}}{2^{k_n}} = \frac{1}{2^n}$
   $= \mathbb{P}(d_{k_1} = \omega_1)\mathbb{P}(d_{k_2} = \omega_2)\ldots\mathbb{P}(d_{k_n} = \omega_n)$.
   Thus, $\{d_k\}$ are i.i.d $Ber(\frac{1}{2})$ random variables.

   (b) Show that $U =: \sum_{k=1}^{\infty} \frac{d_k}{2^k}$ is a Uniform $[0,1]$ random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.
   Solution : $\sum_{k=1}^{\infty} \frac{d_k}{2^k} = x$ iff $0.d_1d_2\ldots$ is the binary expansion of x [with $x \in [0,1]$].
   Thus, $\mathbb{P}(U \in (a,b))$ $=\mathbb{P}((a,b))$$=b-a[\because \sum_{k=1}^{\infty} \frac{d_k(\infty)}{2^k} = \omega]$
   Thus, U is a uniform random variable.

   (c) Let $U_k = \sum_{m=1}^{\infty} \frac{d_{2^{k-1}(2m-1)}}{2^m}$. Show that $\{U_k\}_{k\geq 1}$ are an i.i.d. sequence of Uniform random variables.
   Solution : Now if $A_k = \{2^{k-1}(2m-1) | m \in \mathbb{N}\}$

Then $A_1, A_2, \ldots$ are disjoint subsets of $\mathbb{N}$ and so that random variables that make up U_k for variables k are all independent of each other and hence $\{U_k\}_{k\geq 1}$ are independent.
Again in 6.(b) we saw that if $\{b_k\}_{k\geq 1}$ is an i.i.d. $Ber(\frac{1}{2})$ sequence then $\sum_{k\geq 1} \frac{b_k}{2^k}$ is $Unif(0,1)$ random variable.
Thus, $U_k \sim Unif(0,1)$ $\forall k \geq 1$.

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