Due Date: 24th August 2022, 10pm *Problems Due: 3(a), 4,6*

1. Construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and events $\{A_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ but $\mathbb{P}(A_n \text{ occurs i.o. }) < 1$.

2. Let $\{X_n\}_{n\geq 1}$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that
$$\mathbb{P}(\lim_{n\to\infty} X_n = 0) = 1$$
if and only if
$$\sum_{n=1}^{\infty} \mathbb{P}(\mid X_n \mid \geq \epsilon) < \infty, \text{ for all } \epsilon > 0.$$

3. Let $\{X_n\}_{n\geq 1}$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.
   (a) Suppose $X_n \sim \text{Normal}(0,1)$ then show that $\mathbb{P}(\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log(n)}} = 1) = 1$.
   (b) Suppose $X_n \sim \text{Poisson}(1)$ then show that $\mathbb{P}(\limsup_{n\to\infty} X_n \frac{\log\log(n)}{\log(n)} = 1) = 1$.

4. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Suppose $\Omega = \{-1,1\}^{\mathbb{N}}$ along with
$$\sigma(\{\pi_n^{-1}(E) : E \subseteq \pi_n(\Omega), n \geq 1\})$$
where $\pi_n : \Omega \to \Omega_n$ denote the projection onto the first n coordinates. Suppose $\mathbb{P} : \mathcal{A} \to [0,1]$ given by
$$\mathbb{P}(\{\omega \in \Omega | \pi_n(\omega) = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n)\}) = \frac{1}{2^n},$$
for any $(\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n) \in \{-1,1\}^n$. Show that
$$\mathbb{P}(\{\omega \in \Omega | \sum_{n=1}^{\infty} \omega_n a_n < \infty\}) \in \{0,1\}.$$

5. Let F be a distribution function, namely:
F is monotonically non-decreasing - i.e. $x \leq y \Rightarrow F(x) \leq F(y)$,
F is right continuous - i.e., $\lim_{y\downarrow x} F(y) = F(x)$,
and F satisfies $\lim_{x\to\infty} F(x) = 1, \lim_{x\to -\infty} F(x) = 0$.
Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}_{[0,1]}$ the Borel-σ algebra on $[0,1]$, and $\mathbb{P}(dw)$ be the Lebesgue measure. Show that there exists a random variable $X : \Omega \to \mathbb{R}$ such that $\mathbb{P}(X \leq x) = F(x)$.

6. (*Constructing independent and identically distributed (i.i.d.) sequence of Uniform random variables*) Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}_{[0,1]}$ the Borel-σ algebra on $[0,1]$, and $\mathbb{P}(dw)$ be the Lebesgue measure. Consider random variables $d_k : [0,1] \to \mathbb{R}$ with $d_k(\omega)$ being the k-th digit in the binary expansion of ω.
   (a) Show that $\{d_k\}_{k\geq 1}$ is an i.i.d. Bernoulli $(\frac{1}{2})$ sequence, i.e.

i. Show that $\mathbb{P}(d_k = 0) = \mathbb{P}(d_k = 1) = \frac{1}{2}$ for all $k \geq 1$.

ii. Show that $\{d_k\}_{k\geq 1}$ is an independent collection of random variables.

(b) Show that $U =: \sum_{k=1}^{\infty} \frac{d_k}{2^k}$ is a Uniform $[0,1]$ random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

(c) Let $U_k = \sum_{m=1}^{\infty} \frac{d_{2^{k-1}(2m-1)}}{2^m}$. Show that $\{U_k\}_{k\geq 1}$ are an i.i.d. sequence of Uniform random variables.

7. (*Constructing i.i.d. sequence of random variables*) Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}_{[0,1]}$ the Borel-σ algebra on $[0,1]$, and $\mathbb{P}(dw)$ be the Lebesgue measure. Suppose we are given a probability distribution $\mathbb{Q}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Construct a sequence $\{X_n\}_{n\geq 1}$ of random variables on $)\Omega, \mathcal{F}, \mathbb{P})$ such

(a) $\{X_n\}_{n\geq 1}$ are independent.

(b) $\mathbb{P}(X_n \in A) = \mathbb{Q}(A)$, for all $A \in \mathcal{B}_{\mathbb{R}}$

References

[Ros06] Jeffrey S Rosenthal. *First Look At Rigorous Probability Theory, A*. World Scientific Publishing Company, 2006.