2. Let $\Omega = \{1,2,3,4\}$, with $\mathcal{F}$ the collection of all subsets of Ω. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $\mathcal{F}$, such that $\mathbb{P}\{1\} = \mathbb{P}\{2\} = \mathbb{P}\{3\} = \mathbb{P}\{4\} = \frac{1}{4}$ and $\mathbb{Q}\{2\} = \mathbb{Q}\{4\} = \frac{1}{2}$ extends to $\mathcal{F}$ by linearity. Finally, let $\mathcal{J} = \{\varnothing, \Omega, \{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}$.

   a. Prove that $\mathbb{P}(A) = \mathbb{Q}(A)$ for all $A \in \mathcal{J}$.
   Solution : $\mathbb{P}\{1\} = \mathbb{P}\{2\} = \mathbb{P}\{3\} = \mathbb{P}\{4\} = \frac{1}{4}$ and $\mathbb{Q}\{2\} = \mathbb{Q}\{4\} = \frac{1}{2}$.
   Thus $\mathbb{Q}\{2,4\} = \frac{1}{2} + \frac{1}{2} = 1$
   $\Rightarrow \mathbb{Q}(\Omega - \{2,4\}) = 1 - 1 = 0$
   $\Rightarrow \{1\} = \mathbb{Q}\{3\} = 0$.
   $\mathcal{J} = \{\phi, \Omega, \{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}$
   $\mathbb{P}(\varnothing) = \mathbb{Q}(\varnothing) = 0$ and $\mathbb{P}(\Omega) = \mathbb{Q}(\Omega) = 1$
   $\mathbb{P}\{1,2\} = \mathbb{P}\{1\} + \mathbb{P}\{2\} = \frac{1}{2}$ and $\mathbb{Q}\{1,2\} = \mathbb{Q}\{1\} + \mathbb{Q}\{2\} = \frac{1}{2}$.
   $\mathbb{P}\{2,3\} = \mathbb{P}\{2\} + \mathbb{P}\{3\} = \frac{1}{2}$ and $\mathbb{Q}\{2,3\} = \mathbb{Q}\{2\} + \mathbb{Q}\{3\} = \frac{1}{2}$
   $\mathbb{P}\{3,4\} = \mathbb{P}\{3\} + \mathbb{P}\{4\} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
   $\mathbb{Q}\{3,4\} = \mathbb{Q}\{3\} + \mathbb{Q}\{4\} = 0 + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{4}$
   Similarly, $\mathbb{P}\{1,4\} = Q\{1,4\} = \frac{1}{2}$
   Thus, $\mathbb{P}(A) = Q(A) = \frac{1}{2}\ \forall A \in \mathcal{J}$.

   b. Prove that there is $A \in \sigma(\mathcal{J})$ with $\mathbb{P}(A) \neq \mathbb{Q}(A)$.
   Solution : Obviously, $\{1,2,3\} = \{1,2\} \cup \{2,3\} \in \sigma(\mathcal{J})$
   But $\mathbb{P}\{1,2,3\} = \frac{3}{4}$ whereas $\mathbb{Q}(\{1,2,3\} = \frac{1}{2}$.
   Thus, $\mathbb{P}\{1,2,3\} \neq \mathbb{Q}\{1,2,3\}$.

   c. Why does this not contradict Proposition 2.5.8?
   Solution : $\mathcal{J}$ is not a semi-algebra as it is not closed under finite intersection.
   $\{2,3\} \in \mathcal{J}$ and $\{1,2\} \in \mathcal{J}$.
   But, $\{1,2\} \cap \{2,3\} = \{2\} \notin \mathcal{J}$.
   So it does not contradict 2.5.8.

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the uniform distribution on $\Omega = \{1,2,3\}$, as in Example 2.2.2. Give an example of a sequence $A_1, A_2, \ldots \in \mathcal{F}$ such that

$$\mathbb{P}(\liminf_n A_n) < \liminf_n \mathbb{P}(A_n) < \limsup_n \mathbb{P}(A_n) < \mathbb{P}(\mathop{lim\,sup}_n A_n).$$

   Solution : Let $A_{2k} = \{2,3\}$ and $A_{\{2k-1\}} = \{1\}\ \forall k \in \mathbb{N}$.
   Then $\mathbb{P}(\liminf_n A_n = \mathbb{P}(\cup_n \cap_{k\geq n} A_k) = \mathbb{P}(\varnothing) = 0$.
   $\mathbb{P}(\limsup_n A_n) = \mathbb{P}(\cap_n \cup_{k\geq n} A_k) = \mathbb{P}\{1,2,3\} = 1$.
   But, $\liminf_n \mathbb{P}(A_n) = \frac{1}{3}$ and $\limsup_n \mathbb{P}(A_n) = \frac{2}{3}$. Because,

$$\mathbb{P}(A_n) = \begin{cases} \dfrac{1}{3} & \text{if n is odd} \\ \dfrac{2}{3} & \text{if n is even.} \end{cases}$$

   Thus, $\mathbb{P}(\liminf_n A_n) < \liminf_n \mathbb{P}(A_n) < \limsup_n \mathbb{P}(A_n) < \mathbb{P}(\limsup_n A_n)$.

5. Let X be a random variable with $\mathbb{P}(X > 0) > 0$. Prove that there is $\delta > 0$ such that $\mathbb{P}(X \geq \delta) > 0$.[Hint: Do not forget continuity of probabilities]

   Solution : Let $A_n = \{X \in [\frac{1}{n}, \infty)\}$ then $A_n \subset A_{n+1}$ $\forall n \geq 1$ and so

$$\mathbb{P}(\cup_{n\geq 1} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$$
$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(X > 0) > 0.$$

   Thus, $\exists\ n$ with $\mathbb{P}(A_n) > 0 \Rightarrow \mathbb{P}(X \geq \frac{1}{n}) > 0$.
   Taking $0 \leq \delta \leq \frac{1}{n}$ we get $\mathbb{P}(X \geq \delta) > 0$.