4. Let $I = [0, 1]$. Let $I_1 = I_{11} = (\frac{1}{3}, \frac{2}{3})$ be the open middle third interval of I. Next, let $I_{21} = (\frac{1}{9}, \frac{2}{9})$ and $I_{22} = (\frac{7}{9}, \frac{8}{9})$ be the two open middle third intervals of $I - I_1$. Let $I_2 = I_{21} \cup I_{22}$. For $j \geq 3$ and $k = 1, 2, 3 \ldots, 2^{j-1}$, let I_{jk} be the open middle third intervals of $I - \cup_{k=1}^{j-1} I_k$ and let $I_j = \cup_{k=1}^{2^{j-1}} I_{jk}$. Finally, let $C = I - \cup_{j=1}^{\infty} I_j$. C is called the **Cantor set** .

   (a) Show that C is compact and uncountable.

   Solution: Let $C_o = I$ and for $k \geq 1$, $C_k = I - I_k$. Now I_k is open and hence C_k is closed. Moreover $C = \cap_{k\geq 0}^{\infty} C_k$ and hence C is closed as well and being a subset of the compact set I, C is compact. Again C_k closed in comapct set I and hence compact and $C_o \supseteq C_1 \supseteq C_2 \supseteq \ldots$ is a nested sequence of compact sets so that $C = \cap_{k\geq 0}^{\infty} C_k$ is non-empty. We will show that C is a perfect set which proves that C is uncountable.(see Rudin's *Principles of mathematical analysis*)

   To show that C is perfect we take $x \in C$ and show that x can not take be an isolated point of C. Notice that $C_1 = [0, \frac{1}{2}] \cup [\frac{2}{3}, 1], C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ etc so C_k is a union of closed intervals E_{kj} with $1 \leq j \leq 2^k$ of length $\frac{1}{3^k}$ each. Also every end-point of each E_{kj} is contained in C by definition. Thus if S is an open intervalwith $x \in S$ by taking k large enough we can find $E_{kj} \subset S$. Let x_k be an end point of E_{kj} which is not x. Then $x_k \in C \cap S$ and hence x is not an isolated point and so C is perfect.

   (b) Show that $\lambda(C) = 0$, where λ is lebesgue measure on $[0, 1]$.

   Solution: $\lambda(C_K) = \frac{2^k}{3^k}$ and hence given any $\epsilon > 0$, C_k can be covered by open intervals of total length less than $\frac{2^k}{3^k} + \frac{\epsilon}{2}$. Taking k large enough say $K \geq n_o$ make $(\frac{2}{3})^k < \frac{\epsilon}{2}$ and that C_k can be covered by open interval of total length less than ϵ if $k \geq n_o$ as $C \subseteq C_k$ $\forall k \geq 0$, C can be covered by open intervals of total length $< \epsilon$ as well. Thus, $\lambda(C) = 0$.

5. If μ is a probability measure defined on the Borel σ - algebra $\mathcal{A}$ of $\mathbb{R}$, define $F : \mathbb{R} \to [0, 1]$ by $F(x) = \mu((-\infty, x])$, and verify that

   (a) F is monotonically non-decreasing i.e. $x \leq y \Rightarrow F(x) \leq F(y)$ and right continuous i.e., $\lim_{y\downarrow x} F(y) = F(x)$;

   Solution: If $x \leq y \Rightarrow (-\infty, x] \subseteq (-\infty, y]$ and hence by monotonicity of measures we have

$$F(x) = \mu((-\infty, x]) \leq \mu((-\infty, y]) = F(y).$$

   Now suppose $\{x_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = 0$ and $x_n \geq x_{n+1} \forall n$.

   Let $A_n = (-\infty, x + x_n)$ then $A_n \supseteq A_{n+1} \forall n \geq 1$ and $\cap_{n\geq 1}^{\infty} A_n = (-\infty, x]$. Finally as $\mu(A_1) < \infty$ by the continuity of measures we have

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) \Rightarrow \lim_{n\to\infty} F(x + x_n) = F(x).$$

   Thus, F is right continuous.

   (b) F is discontinuous at x if and only if $\mu(\{x\}) > 0$;

   Solution: Suppose $\mu(\{x\}) = 0$. Then consider a sequence $\{x_n\}_{n\geq 1}$ in $\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = 0$ and $x_n \geq x_{n+1}$ and $x_n \neq 0$ $\forall n$.

   Let $B_n = (-\infty, x - x_n]$. Then clearly, $B_n \subseteq B_{n+1}$ and $\bigcup\limits_{n\geq 1} B_n = (-\infty, x)$. Then by the continuity of measure,

$$\lim_{n\to\infty} \mu(B_n) = \mu((-\infty, x)) = \mu((-\infty, x]) \qquad [\because \mu(\{x\} = 0]$$

This implies,

$$\lim_{n\to\infty} F(x - x_n) = F(x).$$

But then F is left-continuous as well and hence continuous. But if $\mu(\{x\}) > 0$ then clearly, $\lim_{n\to\infty} \mu(B_n) \neq \mu((-\infty, x])$ and hence for this sequence $\{x_n\}_{n\geq 1}$,

$$\lim_{n\to\infty} F(x - x_n) \neq F(x)$$

and so F is discontinuous.

(c) $\lim_{x\to\infty} F(x) = 1, \lim_{x\to-\infty} F(x) = 0$.

Solution: Let $\{x_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = \infty$ and $x_{n+1} \geq x_n\ \forall n$. Then if $A_n = (-\infty, x_n] \Rightarrow A_n \subseteq A_{n+1}\ \forall n$. Thus by the continuity of measure,

$$\lim_{n\to\infty} \mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n) = \mu(\mathbb{R}) = 1.$$

This implies,

$$\lim_{n\to\infty} F(x_n) = 1.$$

Since the above happens for all sequences x_n with $x_n \leq x_{n+1}\ \forall n$ and $\lim_{n\to\infty} x_n = \infty$ we have

$$\lim_{n\to\infty} F(x) = 1.$$

Let $\{y_n\}$ be a sequence with $\lim_{n\to\infty} y_n = -\infty$ and $y_{n+1} \leq y_n\ \forall n$. Let $B_n = (-\infty, y_n]$ clearly $B_n \supseteq B_{n+1}\ \forall n$ and $\mu(B_1) < \infty$, by the continuity of measure,

$$\lim_{n\to\infty} \mu(B_n) = \mu(\cap_{n\geq 1}^{\infty} B_n) = \mu(\varnothing) = 0$$

This implies,

$$\lim_{n\to\infty} F(y_n) = 0.$$

Since the above happens for all sequences $\{y_n\}$ with $\lim_{n\to\infty} y_n = -\infty$ and $y_{n+1} \leq y_n\ \forall n$ we have,

$$\lim_{x\to-\infty} F(x) = 0.$$

(d) Conversely, if $F : \mathbb{R} \to [0, 1]$ is a function satisfying (a), (b) and (c) above, show that there exists a unique probability measure μ on $\mathbb{R}$ such that $\mu((-\infty, x]) = F(x)$ for all x in $\mathbb{R}$.

Solution: Let S be the set of all half open intervals $(a, b]$ and intervals of form $(-\infty, a]$ on the ral line along with $\varnothing$ and define the set function μ on S as:

$$\begin{aligned} \mu((a, b]) &= F(b) - F(a) \\ \mu((-\infty, a]) &= F(a) \\ \text{and} \quad \mu(\varnothing) &= 0 \end{aligned}$$

Clearly μ is countably monotone and finitely additive and hence a probability measure on S. Also S is a semi-ring of subsets of $\mathbb{R}$.

Thus by the Caratheodary-Hahn theorem μ extends to an outer measure $\bar{\mu}$ on the subsets of $\mathbb{R}$ and the $\bar{\mu}$ measurable subsets are exactly the Borel $\sigma-$algebra generated by the open intervals in $\mathbb{R}$.

Finally, $\bar{\mu}$ restricted to the Borel $\sigma-$algebra is a probability measure as,

$$\lim_{n\to\infty} \bar{\mu}((-\infty, n]) = \bar{\mu}(\mathbb{R})$$

This implies,

$$\bar{\mu}(\mathbb{R}) = \lim_{n\to\infty} F(n) = 1.$$

The function F is referred to as the **distribution function** of μ.