Due Date: November 9th, 2022, 10pm
Problems Due: 1,3,9

1. Let $\mathbb{P}$ be a Probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Let F be a closed set and for $\delta > 0$ let $F_\delta = \{x \in \mathbb{R} : d(x, F) \leq \delta\}$. Show that $\{x \in \mathbb{R} : d(x, F) = \delta\} \supset \partial F_\delta$ and are disjoint for distinct $\delta > 0$. Further show that there are only countably many $\delta > 0$ such that $\mathbb{P}(\{x \in \mathbb{R} : d(x, F) = \delta\}) > 0$.

2. Show that for $n \geq 1$ and $u > 0$
$$\int_0^n \sin(s) \exp(-us) ds = \frac{1 + \beta_n - u\alpha_n}{1 + u^2},$$
where $\alpha_n, \beta_n \to 0$ as $n \to \infty$ and using the above with Fubini's Theorem show that
$$\int_0^n \frac{\sin(s)}{s} ds \to \frac{\pi}{2} \text{ as } n \to \infty.$$

3. Complete the proof of Skorokhod theorem done in class. I.e. Show that
$$\limsup_{n\to\infty} Y_n(\omega) \leq Y(\omega),$$
for all continuity points ω of Y and show that Y has only countably many discontinuities.

4. Prove Lemma 11.1.8 in [Ros06]

5. Let $\mathbb{P}, \{\mathbb{P}_n\}_{n\geq1}$ be Probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Suppose that for every subsequence $\mathbb{P}_{n_k}$ there is a further subsequence $\mathbb{P}_{n_{k_l}}$ that converges weakly to $\mathbb{P}$. Show that $\mathbb{P}_n$ converge weakly to $\mathbb{P}$.

6. Exercise 5.3.3 in [AS08]

7. Suppose X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[X^2] < \infty$. Show that
$$\mathbb{E}[\min\{\frac{| X |^3}{\sqrt{n}}, X^2\}] \to 0 \text{ as } n \to \infty.$$

8. Suppose $z_n \in \mathbb{C}$ such that $z_n \to z$ as $n \to \infty$ then show that
$$\left(1 + \frac{z_n}{n}\right)^n \to \exp(z) \text{ as } n \to \infty.$$

9. Let $\{X_n\}_{n\geq1}, X$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that $X_n \xrightarrow{p} X$ implies that $X_n \xrightarrow{d} X$ as $n \to \infty$.

References

[AS08] Siva Athreya and Viakalathur Shankar Sunder. *Measure & probability.* Universities Press, 2008.

[Ros06] Jeffrey S Rosenthal. *First Look At Rigorous Probability Theory, A.* World Scientific Publishing Company, 2006.

C-12 Probability Theory **November 3, 2022** **Quiz 10** **Score**

Name ____________________________ **Seat**

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1. Let X_n be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X_n = (-1)^n Y$ where $Y \sim$ Normal $(0, 1)$ random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Decide if: X_n converges almost surely or in Probability or in Distribution ?