Due Date: October 26th 2022, 10pm
Problems Due: 1 b(iii), 3(a), 3(b), 6

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. Consider two independent random variables X, Y on $(\Omega, \mathcal{F}, \mathbb{P})$.
   (a) Show that the distribution of $X+Y$ is given by the convolution of the distributions of X and Y. That is if $\mathbb{Q}(\cdot) = \mathbb{P}\circ(X+Y)^{-1}(\cdot)$, $\mathbb{P}_X(\cdot) = \mathbb{P}\circ X^{-1}(\cdot)$ and $\mathbb{P}_Y(\cdot) = \mathbb{P}\circ Y^{-1}(\cdot)$ then
   $$\mathbb{Q}(A) = \int_{\mathbb{R}} \mathbb{P}_X(A-y)d\mathbb{P}_Y(dy),$$
   for all Borel sets A, with $A - y = \{z \in \mathbb{R} : z + y \in A\}$.
   (b) Compute the distribution of $X + Y$ when
      i. $X \sim$ Poisson (λ) and $Y \sim$ Poisson (μ)
      ii. $X \sim$ Normal (a, u) and $Y \sim$ Normal (b, v)
      iii. $X \sim$ Gamma (α, β) and $Y \sim$ Gamma(η, β)
2. Let a closed set F and $\epsilon > 0$ be given. Show that there is a function uniformly continuous $f : \mathbb{R} \to [0, 1]$ such that
   $$f(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } \mathrm{d}(x, F) > \epsilon \end{cases}$$
3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. Suppose $\{X_n\}_{n\geq 1}$, X are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X_n \xrightarrow{d} X$ or X_n converges in distribution to X (i.e. $\mathbb{P}\circ X_n^{-1}$ converges weakly to $\mathbb{P}\circ X^{-1}$).
   (a) Ex 5.2.11 in [AS08]
   (b) Ex 5.2.19 in [AS08]
   (c) Ex 5.2.23 in [AS08]
4. Ex 10.3.1 in [Ros06]
5. Ex 10.3.3 in [Ros06]
6. Ex 10.3.4 in [Ros06]

References

[AS08] Siva Athreya and Viakalathur Shankar Sunder. *Measure & probability*. Universities Press, 2008.

[Ros06] Jeffrey S Rosenthal. *First Look At Rigorous Probability Theory, A*. World Scientific Publishing Company, 2006.