C-12 Probability Theory **August 25th, 2022** **Quiz 3** **Score**

Name Solution **Seat**

1. Let $\{X_n\}_{n\geq 1}$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that

$$\sum_{n=1}^{\infty} \mathbb{P}(\mid X_n - 5 \mid \geq \epsilon) < \infty, \text{ for all } \epsilon > 0.$$

implies that

$$\mathbb{P}(\lim_{n\to\infty} X_n = 5) = 1$$

let $k \geq 1$ and $\varepsilon = \frac{1}{k}$ be given.

let $A_n^k = \{ \mid |X_n - 5| \geq \frac{1}{k} \}$ $\forall n \geq 1$

By assumption $\sum_{n=1}^{\infty} \mathbb{P}(A_n^k) < \infty$

$\therefore$ By Borel-Cantelli lemma

$$\mathbb{P}(\limsup_{n\to\infty} A_n^k) = 0$$

i.e. $\mathbb{P}(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m^k) = 0.$

As $k \geq 1$ was arbitrary we have that

$$\mathbb{P}(\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m^k) \leq \sum_{k=1} \mathbb{P}(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m^k) = 0.$$

$$\Rightarrow \mathbb{P}(\bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} (A_m^k)^c) = 1$$

$$\Rightarrow \mathbb{P}(\forall k \geq 1 \ \exists n \geq 1 : |X_m - 5| \leq \frac{1}{k} \ \forall m \geq k) = 1$$

$$\Rightarrow \mathbb{P}(\lim_{n\to\infty} X_n = 5) = 1 \qquad \square$$