1 F is a closed set. Let $\delta > 0$.

$F_\delta = \{x \in \mathbb{R} : d(x, F) \leq \delta\}$.

Let $x \in \partial F_\delta$, then $\exists$ a sequence $\{z_n\}_{n\in\mathbb{N}}$ in F_δ such that $z_n \to x$ as $n \to \infty$.

Let $y \in F$, and $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $d(x, z_n) < \epsilon\ \forall n \geq N$.

Now for $n \geq N$,

$$d(x, y) \leq d(x, z_n) + d(z_n, y) < \epsilon + \delta$$

.

Since $\epsilon > 0$ was arbitrary, we have $d(x, y) \leq \delta$.

Since $y \in F$ was arbitrary, we have,

$$\begin{aligned} & d(x, y) \leq \delta \quad \forall y \in F \\ \Rightarrow & d(x, F) = \inf\{d(x, y) : y \in F\} \leq \delta \\ \Rightarrow & x \in F_\delta \end{aligned}$$

Now suppose $d(x, F) < \delta$, then $\exists\ y_0 \in F$ such that $d(x, y_0) = a < \delta$.

Define

$$U = \{z \in \mathbb{R} : d(x, z) < p\} \text{ where } 0 < p < \delta - a$$

then U is a open set around x.

Let $z \in U$, then

$$d(z, y) \leq d(z, x) + d(x, y) < p + a < \delta.$$

This implies $d(z, F) \leq \delta$.

Since $z \in U$ was arbitrary, this implies $U \subset F_\delta$, which contradicts the fact that $x \in \partial F_\delta$.

Therefore for $x \in \partial F_\delta$ we must have $d(x, F) = \delta$.

So,

$$\partial F_\delta \subset \{x \in \mathbb{R} : d(x, F) = \delta\}.$$

Now suppose there are uncountably many $\delta's$ such that

$$\mathbb{P}(\{x \in \mathbb{R} : d(x, F) = \delta\}) > 0.$$

Define

$$E^\delta = \{x \in \mathbb{R} : d(x, F) = \delta\}$$

and

$$A = \{\delta > 0 : \mathbb{P}(E^\delta) > 0\}.$$

$E^{\delta_1} \cap E^{\delta_2} = \phi$ since $d(x, F)$ cannot take both values δ_1 and δ_2 by definition.

Define

$$A_n = \{\delta \in A : \mathbb{P}(E^\delta) > \frac{1}{n}\}.$$

Let $\delta_1, \delta_2, \ldots, \delta_k \in A_n$ then

$$\mathbb{P}(E^{\delta_1} \cup E^{\delta_2} \cup \ldots \cup E^{\delta_k}) = \sum_{i=1}^{k} \mathbb{P}(E^{\delta_i}) > \frac{k}{n}.$$

Since $\mathbb{P}(\mathbb{R}) = 1 \Rightarrow A_n$ can have atmost $n-1$ elements.

Then $A_n \subset A$ and for any $\delta \in A$, $\exists\ N$ such that $\mathbb{P}(E^{\delta}) > \frac{1}{N}$ which implies that $\delta \in A_N$ for some $N \in \mathbb{N}$.

So,

$$A = \underset{n \in \mathbb{N}}{\cup} A_n$$

.

Each A_n has finitely many elements and the union of countably many finite sets is countable. Hence A is countable.

3. Let $\{X_n\}_{n \geq 1}$ is a sequence of random variables such that $X_n \xrightarrow{d} X$.

   We write $F_n = F_{X_n}$ as the cdf of X_n. And, $F = F_X$ as the cdf of X.

   We have $\Omega = [0,1], \mathcal{B} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} =$ Lebesgue measure on [0,1].

$$Y_n(\omega) = \inf\{x \in \mathbb{R} : \omega \leq F_n(x)\}$$

and

$$Y(\omega) = \inf\{x \in \mathbb{R} : \omega \leq F(x)\}.$$

We know that $F_Y = F$ and $F_{Y_n} = F_n$.

Let $\omega \in \Omega$ and let $\omega_0 \in \Omega$ be such that $\omega < \omega_0$.

Let $\delta > 0$ be such that $b = Y(\omega_0) + \delta$ is a continuity point of F.

So

$$\begin{aligned} Y(\omega_0) < b \Rightarrow F(b) \geq \omega_0 & \\ & \Rightarrow \exists m_2 \in \mathbb{N} \text{ such that } F_n(b) \geq w_0 - \delta \quad \forall n \geq m_2 [\text{ because } X_n \xrightarrow{d} X] \\ & \Rightarrow \exists m_2 \in \mathbb{N} \text{ such that } Y_n(\omega_0 - \delta) \leq b \quad \forall n \geq m_2 \end{aligned}$$

Then we have,

$$\limsup_{n \to \infty} Y_n(\omega_0 - \delta) \leq b = Y(\omega_0) + \delta \tag{1}$$

We further restrict δ to the interval $(0, \omega_0 - \omega)$.

For any $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 < \omega_2$, if x satisfies $F_n(x) \geq \omega_2$ then $F_n(x) \geq \omega_1$.

Thus implies, $Y_n(\omega_1) \leq Y_n(\omega_2)$. So Y_n is increasing function on Ω.

From (1) and using the fact that $\delta < \omega_0 - \omega$ and Y_n's are increasing function on Ω we get,

$$\limsup_{n \to \infty} Y_n(\omega) \leq Y(\omega_0) + \delta$$

.

Now we take a sequence of δ_n in the interval $(0, \omega_0 - \omega)$ such that

$Y(\omega_0) + \delta_n$ were continuity points of F and $\delta \to 0$ as $n \to \infty$, then

$$\limsup_{n\to\infty} Y_n(\omega) \leq Y(\omega_0).$$

We have established that for any $\omega, \omega_0 \in \Omega$ such that $\omega < \omega_0$,
the above results holds.
If ω is a continuity point of Y, we then allow ω_0 to decrease to ω,
we get

$$\limsup_{n\to\infty} Y_n(\omega) \leq Y(\omega)$$

.
Forany $\omega_1, \omega_2 \in \Omega$ with $\omega_1 < omega_2$ if x satisfies $\omega_2 \leq F(x)$ then $\omega_1 \leq F(x)$.
So,

$$\{x \in \mathbb{R} : \omega_1 \leq F(x)\} \subset \{\mu \in \mathbb{R} : \omega_2 \leq F(x)\}.$$

Taking infimum on both side, we get,

$$Y(\omega_1) \leq Y(\omega_2).$$

So, Y is increasing function.
This implies, Y has countably many discontinuities.

9. $\{X_n\}_{n\geq 1}$, X are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \xrightarrow{p} X$ as $n \to \infty$.
Let $\epsilon > 0$, let for $t \in \mathbb{R}$, $F_{X_n}(t) = \mathbb{P}(X_n \leq t)$ and $F_X(t) = \mathbb{P}(X \leq t)$.

$$\begin{aligned}
F_{X_n}(t) &= \mathbb{P}(X_n \leq t) \\
&= \mathbb{P}(X_n \leq t, |X_n - X| > \epsilon) + \mathbb{P}(X_n \leq t, |X_n - X| \leq \epsilon) \\
&\leq \mathbb{P}(|X_n - X| > \epsilon) + \mathbb{P}(X_n \leq t, |X_n - X| \leq \epsilon) \\
&\leq \mathbb{P}(|X_n - X| > \epsilon) + \mathbb{P}(X \leq t + \epsilon) \\
&= \mathbb{P}(|X_n - X| > \epsilon) + F_X(t + \epsilon) \qquad (2)
\end{aligned}$$

Similarly we can show,

$$F_X(t - \epsilon) \leq \mathbb{P}(|X_n - X| \geq \epsilon) + F_{X_n}(t).$$

Then,

$$F_{X_n}(t) \geq F_X(t - \epsilon) - \mathbb{P}(|X_n - X| \geq \epsilon. \qquad (3)$$

From (2) we get,

$$\limsup_{n\to\infty} F_{X_n}(t) \leq \limsup_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) + F_X(t + \epsilon)$$

and from (3) we get,

$$\liminf_{n\to\infty} F_{X_n}(t) \geq F_X(t - \epsilon) - \liminf_{n\to\infty} \mathbb{P}(|X_n - X| \geq \epsilon).$$

We have, $X_n \xrightarrow{p} X$ as $n \to \infty$.

This implies,

$$\limsup_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0 \text{ and } \liminf_{n\to\infty} \mathbb{P}(|X_n - X| \geq \epsilon) = 0.$$

So we get,

$$\limsup_{n\to\infty} F_{X_n}(t) \leq F_X(t+\epsilon) \text{ and } \liminf_{n\to\infty} F_{X_n}(t) \geq F_X(t-\epsilon).$$

If t is continuity point of F, then letting $\epsilon \to 0$, we get,

$$\limsup_{n\to\infty} F_{X_n}(t) \leq F_X(t) \leq \liminf_{n\to\infty} F_{X_n}(t).$$

This implies,

$$\lim_{n\to\infty} F_{X_n}(t) = F_X(t)$$

when t is a continuity point.