1. Exercise 11.5.1 in [**?**]

   Solution 1(a):

   Let $\mu_n = s_n$ are point masses for $n \geq 1$.

   The sequence $\{\mu_n\}$ is not tight(roughly speaking the point masses are going to infinity).

   Take $\epsilon = \frac{1}{2}$. There exists any $a < b, a, b \in \mathbb{R}$ such that $\mu_n([a, b]) \geq 1 - \epsilon = \frac{1}{2}$ for all $n \in \mathbb{N}$.

   Take any $n_0 > b + 1$, this criteria will break. So the sequence is not tight.

   Solution 1(b): Since the point masses are going to infinity, every subsequence of $\{\mu_n\}$ are also going to infinity. So $\nexists$any Borel probability measure μ such that $\mu_{n_k} \Rightarrow \mu$.

   The theorem in this section are related to tightness of sequences. Since $\{\mu_n\}$ is tight, no theorems will apply here.'

   Solution 1(c):

   $$F_n(x) = \mu_n((-\infty, x])$$

   . Choose $F(x) = 0 \ \forall x \in \mathbb{R}$.

   Clearly F is right continuous and

   $$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \mu_n((-\infty, x]) = 0 = F(x) \quad \forall x.$$

   $\{F_n\}$ is a sequence of cumulative distribution here. Then by Helly Selection pronciple $\exists$ a subsequence $\{F_{n_k}\}$ and a non-decreasing right continuous function F such that F is continuous at x.

   F is identically zero here.

   Solution 1(d): Take $\mu_n = s_n$. Then

   $$F_n(x) = \mu_n((-\infty, x])$$

   .

   Choose $F(x) = 1$. Then,

   $$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \mu_n((-\infty, x]) = 1 = F(x).$$

   Clearly F is right continuous and Helly Selection principle applies here.

3(a) Exercice 11.5.7 in [**?**].

Soluiton 3(a): The Borel probability measure μ_n is defined by $\mu_n(\{x\}) = \frac{1}{n}$ for $x = 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$.

λ is the Lebesgue measure on $[0, 1]$.

$$\begin{aligned}
\phi_n(t) &= \int e^{itx} \mu_n(dx) \\
&= \sum_{x=0}^{n-1} e^{\frac{itx}{n}} \mu_n(\frac{x}{n}) \\
&= \frac{1}{n} \sum_{x=0}^{n-1} e^{\frac{itx}{n}} \\
&= \frac{1 - e^{it}}{n(1 - e^{\frac{it}{n}})}
\end{aligned}$$

Soluiton 3(b):

$$\begin{aligned}\phi(t) &= \int e^{itx}\lambda(dx)\\ &= \int e^{itx}dx\\ &= \frac{e^{itx}}{it}|_0^t\\ &= \frac{1-e^{it}}{-it}\end{aligned}$$

Soluiton 3(c):

$$\lim_{n\to\infty}\phi_n(t) = (1-e^{it})\lim_{n\to\infty}\frac{1}{n(1-e^{\frac{it}{n}})}$$

$$\lim_{n\to\infty} n(1-e^{\frac{it}{n}}) = \lim_{h\to 0}\frac{1-e^{ith}}{h} = -it \quad \text{[By L-Hospital rule]}.$$

Then,

$$\lim_{n\to\infty}\phi_n(t) = \frac{1-e^{it}}{-it}$$

.

So,

$$\phi_n(t) \to \phi(t) \quad \forall t \in \mathbb{R}.$$

Soluiton 3(d): By continuity theorem, if $\lambda, \mu_1, \mu_2, \ldots$ are probability measures with corresponding charesteristic functions $\phi, \phi_1, \phi_2, \ldots$, then $\mu_n \Rightarrow \lambda$ iff $\phi_n(t) \to \phi(t)$ $\forall t \in \mathbb{R}$.

Since $\phi_n(t) \to \phi(t)$ $\forall t \in \mathbb{R}$ by (c), we have μ_n converges weakly to λ, the uniform distribution on $[0, 1]$.

5. Exercise 5.1.8 in [**?**].

Solution:

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itx}\phi_x(t)dt$$

given that $\int_{\mathbb{R}}|\phi_X(t)|dt < \infty$.

$$\begin{aligned}|f(x)| &= \frac{1}{2\pi}|\int_{\mathbb{R}} e^{itx}\phi_X(t)|\\ &\leq \frac{1}{2\pi}\int_{\mathbb{R}}|e^{itx}||\phi_X(t)|dt\text{[By Triangle inequality]}\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}|\phi_X(t)|dt < \infty\end{aligned}$$

So f is bounded on $\mathbb{R}$.

Next,

$$
\begin{aligned}
|f(x+h) - f(x)| &= \frac{1}{2\pi}|\int_{\mathbb{R}} e^{it(x+h)}\phi_X(t)dt - \int_{\mathbb{R}} e^{itx}\phi_X(t)dt| \\
&\leq \frac{1}{2\pi}\int_{\mathbb{R}} |\phi_X(t)|e^{itx}|(e^{ith} - 1|dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\phi_X(t)||(e^{ith} - 1)|dt \\
&\leq |(e^{ith} - 1)|\frac{1}{2\pi}\int_{\mathbb{R}} |\phi_X(t)|dt
\end{aligned}
$$

Let

$$
M = \frac{1}{2\pi}\int_{\mathbb{R}} |\phi_X(t)|dt < \infty.
$$

Since e^{itx} is continuous, $\forall \epsilon > 0$, $\exists\ \delta > 0$ such that when $|h| < \delta$, $|e^{ith} - 1| < \frac{\epsilon}{2M}$.

$$
\Rightarrow |f(x+h) - f(x)| < \epsilon
$$

for $|h| < \delta$.

Since $\epsilon > 0$ is arbitrary, we have f is continous.