1. Consider two independent random variables X, Y on $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Show that the distribution of $X+Y$ is given by the convolution of the distributions of X and Y. That is if $\mathbb{Q}(\cdot) = \mathbb{P} \circ (X+Y)^{-1}(\cdot)$, $\mathbb{P}_X(\cdot) = \mathbb{P} \circ X^{-1}(\cdot)$ and $\mathbb{P}_Y(\cdot) = \mathbb{P} \circ Y^{-1}(\cdot)$ then

$$\mathbb{Q}(A) = \int_{\mathbb{R}} \mathbb{P}_X(A-y) d\mathbb{P}_Y(dy),$$

   for all Borel sets A, with $A - y = \{z \in \mathbb{R} : z + y \in A\}$.

   (b) Compute the distribution of $X+Y$ when

      i. $X \sim$ Poisson (λ) and $Y \sim$ Poisson (μ)

      ii. $X \sim$ Normal (a, u) and $Y \sim$ Normal (b, v)

      iii. $X \sim$ Gamma (α, β) and $Y \sim$ Gamma(η, β)

Solution 1(b)(iii): We have to compute the joint distribution of $X+Y$ when $X \sim Gamma(\alpha, \beta)$ and $Y \sim Gamma(\eta, \beta)$.

The pdf of $Gamma(\alpha, \beta)$ is

$$\mathbb{P}_X(x) = \frac{\beta^\alpha e^{-\beta x} x^{\alpha-1}}{\Gamma(\alpha)}, \quad x > 0, \alpha, \beta > 0$$

and

$$\mathbb{P}_Y(y) = \frac{\beta^\eta e^{-\beta y} y^{\eta-1}}{\Gamma(\eta)}, \quad y > 0, \beta, \eta > 0.$$

Let $Z = X + Y$.

$$\begin{aligned}
\mathbb{P}_Z(z) &= \mathbb{P}_{(X+Y)}(z) \\
&= \int_0^z \mathbb{P}(X+Y=z)\mathbb{P}(dz) \\
&= \int_0^z \mathbb{P}_X(x)\mathbb{P}_y(z-x)dx \\
&= \int_0^z \frac{x^{\alpha-1}e^{-\beta x}\beta^\alpha}{\Gamma(\alpha)} . \frac{(z-x)^{\eta-1}e^{-\beta(z-x)}\beta^\eta}{\Gamma(\eta)} dx \\
&= \frac{\beta^{\alpha+\eta}e^{-\beta z}}{\Gamma(\alpha)\Gamma(\eta)} \int_0^z x^{\alpha-1}(z-x)^{\eta-1}dx
\end{aligned}$$

To evaluate the integral $\int_0^z x^{\alpha-1}(z-x)^{\eta-1}dx$, we will use Gamma function integral:

$$\begin{aligned}
\int_0^z x^{\alpha-1}(z-x)^{\eta-1}dx &= \int_0^1 t^{\alpha-1}z^{\alpha-1}z^{\eta-1}(1-t)^{\eta-1}zdt \quad [x = tz] \\
&= z^{\alpha+\eta-1}\int_0^1 t^{\alpha-1}(1-t)^{\eta-1}dt \\
&= z^{\alpha+\eta-1}\Gamma(\alpha, \eta) \\
&= z^{\alpha+\eta-1}\frac{\Gamma(\alpha)\Gamma(\eta)}{\Gamma(\alpha+\eta)}.
\end{aligned}$$

Therefore,

$$X + Y \sim Gamma(\alpha + \eta, \beta).$$

3. Suppose $\{X_n\}_{n\geq 1}$, X are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X_n \xrightarrow{d} X$ or X_n converges in distribution to X (i.e. $\mathbb{P} \circ X_n^{-1}$ converges weakly to $\mathbb{P} \circ X^{-1}$).

   (a) Ex 5.2.11 in [**?**]

   (b) Ex 5.2.19 in [**?**]

   (c) Ex 5.2.23 in [**?**]

   Solution 3(a):$X_n \xrightarrow{d} X$ then $\mathbb{P}(X_n \leq t) \to \mathbb{P}(X \leq t)$ as $n \to \infty$ $\forall t \in \mathbb{R}$.
   Now, $\mathbb{P}(X_n^2 \leq t) = 0$ if $t < 0$ and $\mathbb{P}(X^2 \leq t) = 0$ if $t < 0$.
   Thus, if $t < 0$ $\mathbb{P}(X_n^2 \leq t) = 0 \to 0 = \mathbb{P}(X^2 \leq t)$.
   If, $t \geq 0$ and $t, \sqrt{t}, -\sqrt{t}$ are points of continuity,

$$\begin{aligned}\mathbb{P}(X_n^2 \leq t) &= \mathbb{X}_{\ltimes} \leq \sqrt{\approx} - \mathbb{P}(X_n \leq -\sqrt{t}) + \mathbb{P}(X_n = -\sqrt{t}) \\ &\to \mathbb{X} \leq \sqrt{\approx} - \mathbb{P}(X \leq -\sqrt{t}) + \mathbb{P}(X = -\sqrt{t}) \text{ as } n \to \infty \\ &= \mathbb{P}(X^2 \leq t).\end{aligned}$$

   Solution 3(b): Since a is a continuity point of F_X, if $\epsilon > 0$ we can select $\delta > 0$ with

$$|F_X(a - \delta) - F_X(a)| < \frac{\epsilon}{3}$$

   such that $a - \delta$ is a continuity point of F_X.
   Now we choose N so that $n \geq N$, this implies,

$$|F_{X_n}(a) - F_X(a)| < \frac{\epsilon}{3} \text{ and } |F_{X_n}(a - \delta) - F_X(a - \delta)| < \frac{\epsilon}{3}.$$

   Thus,

$$\begin{aligned}\mathbb{P}(X_n = a) &\leq |F_{X_n}(a) - F_{X_n}(a - \delta)| \\ &\leq |F_{X_n}(a) - F_X(a)| + |F_X(a) - F_X(a - \delta)| + |F_X(a - \delta) - F_{X_n}(a - \delta)| \\ &< \epsilon\end{aligned}$$

   Thus,

$$\mathbb{P}(X_n = a) \to 0 \text{ as } n \to \infty.$$

6. Ex 10.3.4 in [**?**].

   Solution:
   Suppose $\{\mu_n\}_{n\geq 1} \Rightarrow \mu$ and $\{\mu_n\}_{n\geq 1} \Rightarrow \nu$.
   Then $\forall$ bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$,

$$\int f d\mu_n \to \int f d\mu \quad \text{ as } n \to \infty$$

   and

$$\int f d\mu_n \to \int f d\nu \quad \text{as } n \to \infty.$$

Hence $\forall\ f : \mathbb{R} \to \mathbb{R}$ with f bounded and continous,

$$\int f d\mu = \int f d\nu$$

.

Choose $y \in \mathbb{R}$. Now take a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n \equiv 1$ on $(-\infty, y]$ and $f_n \equiv 0$ on $(y + \frac{1}{n}, \infty)$.

Then

$$\lim_{n\to\infty} f_n = \chi_{(-\infty,y]}$$

.

By dominating convergence theorem,

$$\begin{aligned}
\mu((-\infty, y]) &= \int_{\mathbb{R}} \chi_{(-\infty,y]} d\mu \\
&= \int \lim_{n\to\infty} f_n d\mu \\
&= \lim_{n\to\infty} \int f_n d\mu \\
&= \lim_{n\to\infty} \int f_n d\nu \\
&= \int \lim_{n\to\infty} f_n d\nu \\
&= \int_{\mathbb{R}} \chi_{(-\infty,y]} d\nu \\
&= nu((-\infty, y])
\end{aligned}$$

Therefore $\mu \equiv \nu$ as $\mu(\{y\}) = \nu(\{y\}) = 0$.