Recall :- (Section 12)

Theorem 1 (Monotone Convergence Theorem)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
Let $\{X_n\}_{n\geq 1}$, X be random variables on it.
Suppose $E[X_1] > -\infty$ and $X_n \uparrow X$
Then $E[X_n] \uparrow E[X]$.

Example 3 :- $([0,1], \mathcal{B}_{[0,1]}, d\omega)$

$\forall n \geq 1$ $\quad X_n : [0,1] \to [0,\infty)$ $\qquad X : [0,1] \to \mathbb{R}$

$X_n = n \, \mathbb{1}_{[0,\frac{1}{n}]}$ $\qquad X \equiv 0$

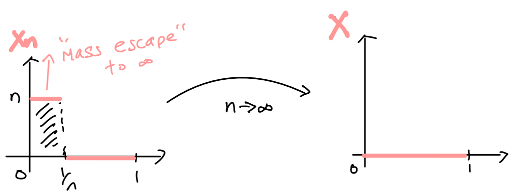
$X_n(\omega) \longrightarrow X(\omega) \quad \forall \, \omega \in \Omega.$

$E[X_n] = n \, \mathbb{P}([0, \tfrac{1}{n}]) = n \cdot \tfrac{1}{n} = 1$

$E[X] = 0$

$\therefore \quad E[X_n] \not\to E[X]$

$\left\{ \begin{array}{l} X_n \leq X_{n+1} \;\; \otimes \\ \text{no contradiction} \\ \text{to M.C.T.} \end{array} \right.$



Reason: $E[X_n] \not\to E[X]$

Theorem 2 (Bounded Convergence Theorem):

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
Let $\{X_n\}_{n\geq 1}$, X be random variables on it.
Suppose - $\exists \, K > 0$ s.t. $|X_n| \leq K \quad \forall \, n \geq 1$
and $X_n \longrightarrow X$ as $n \to \infty$.
Then $E[X_n] \longrightarrow E[X]$ as $n \to \infty$.

Section 15 :- Limit Theorems – II

The next Theorem illustrates that mass "cannot appear at infinity".

Theorem 1 (Fatou's Lemma) : On $(\Omega, \mathcal{F}, \mathbb{P})$ let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that

$$X_n \geq c \quad \forall\, n\geq 1 \quad \text{for some } c\in\mathbb{R}.$$

Then

$$E\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} E[X_n]$$

Proof :- For $n\geq 1$ let $Y_n : \Omega \to \mathbb{R}$ be given by

$$Y_n(\omega) = \inf_{k\geq n} X_k(\omega)$$

(Ex: Y_n are r.v.)

Note : (i) $Y_n \geq c \quad \forall\, n\geq 1$

(ii) $Y_n \leq Y_{n+1} \quad \forall\, n\geq 1$

(iii) $Y_n \longrightarrow \liminf_{n\to\infty} X_n$ as $n\to\infty$

(iv) $Y_n \leq X_n \quad \forall\, n\geq 1$

(i) – (iii) + M.C.T. $\Rightarrow$ $\lim_{n\to\infty} E[Y_n] = E\left[\liminf_{n\to\infty} X_n\right]$ — (1)

(iv) $\Rightarrow$ $E[Y_n] \leq E[X_n] \quad \forall\, n \geq 1$

$$\Rightarrow \liminf_{n\to\infty} E[Y_n] \leq \liminf_{n\to\infty} E[X_n] \qquad - (2)$$

and $\overset{(1)}{\underset{(2)}{\Rightarrow}}$ $E[\liminf_{n\to\infty} X_n] \leq \liminf_{n\to\infty} E[X_n]$

$\square$

Recall [Stated in Week 5]

There are two ways to stop this.

(a) X_n's are bounded $\rightarrow$ [Bounded Convergence Theorem]
Theorem 12.2

(b) X_n's are bounded by an integrable r.v.

We shall demonstrate (b).

Theorem 2 (Dominated Convergence Theorem) On $(\Omega, \mathcal{F}, \mathbb{P})$

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that

- $|X_n| \leq Y \quad \forall\, n \geq 1$ for some integrable random variable Y.

- $X_n \longrightarrow X$ as $n \to \infty$.

Then $E[X_n] \longrightarrow E[X]$ as $n \to \infty$.

Proof:- let $P_n = Y + X_n$ then $P_n \geq 0$

$$P_n \longrightarrow Y + X \quad \text{as} \quad n \to \infty$$

(Fatou) $\Rightarrow$ $E[\liminf_{n\to\infty} P_n] \leq \liminf_{n\to\infty} \mathbb{E}[Y] + E[X_n]$

(Ex.)

$\Rightarrow \quad E[Y] + E[X] \leq E[Y] + \liminf_{n\to\infty} E[X_n]$

$\Rightarrow$

$E[Y]<\infty \quad E[X] \leq \liminf_{n\to\infty} E[X_n] \quad -(3)$

let $M_n = Y - X_n$ then $M_n \geq 0$

$M_n \longrightarrow Y - X$ as $n \to \infty$

(Fatou) $\Rightarrow \quad E[\liminf_{n\to\infty} M_n] \leq \liminf_{n\to\infty} (E[Y] - E[X_n])$

Ex. $\Rightarrow \quad E[Y] - E[X] \leq E[Y] - \limsup_{n\to\infty} E[X_n]$

$\Rightarrow$ $E[Y]<\infty$ $\quad E[X] \geq \limsup_{n\to\infty} E[X_n] \quad -(4)$

(3) $\Rightarrow$ (4) $\quad E[X] = \lim_{n\to\infty} E[X] \quad \square$

© There is another way to control escape of mass to infinity.

Example 1: $(\Omega = [0,1], \mathcal{B}_{[0,1]}, \mathbb{P}(d\omega) = d\omega)$

Let $X_n = \sqrt{n}\, \mathbb{1}_{[0,\frac{1}{n}]}$ and $X = 0$

$X_n \longrightarrow X$ and $E[X_n] = \frac{1}{\sqrt{n}} \longrightarrow 0 = E[X]$

Note: · $X_n(\cdot)$ is not bounded above

(Ex) · $X_n(\cdot)$ is not dominated by integrable Y.

$\Rightarrow$ Something else saves the day!

Fact:- A r.v. X on $(\Omega, \mathcal{F}, P)$ is integrable

$$\Leftrightarrow \quad E[|X|] < \infty$$

$$\boxed{\text{key: } |X| \mathbb{1}_{|X|>k} \uparrow |X| \text{ as } k\to\infty \overset{\text{M.C.T.}}{\Rightarrow} E[|X| \mathbb{1}_{|X|>k}] \uparrow E[|X|] \text{ as } k\to\infty}$$

$$\underset{\text{Ex}}{\Leftrightarrow} \quad E[|X| \mathbb{1}_{|X|>k}] \longrightarrow 0 \text{ as } k \to \infty$$

Now (5) can be used to replace domination by an integrable random variable.

$k \geq 1$ $\quad n \geq 1$

$$E[|X_n| \mathbb{1}_{|X_n|>k}] = \begin{cases} 0 & k > \sqrt{n} \\ \frac{1}{\sqrt{n}} & k \leq \sqrt{n} \end{cases}$$

$$\sup_{n\geq 1} E[|X_n| \mathbb{1}_{|X_n|>k}] = \max\left\{ \frac{1}{\sqrt{n}} \,\middle|\, k \leq \sqrt{n} \right\}$$

$$\leq \frac{1}{k}$$

$\therefore$

$$\sup_{n\geq 1} E[|X_n| \mathbb{1}_{|X_n|>k}] \longrightarrow 0 \text{ as } k \to \infty.$$

The same "k" large works for all $n \geq 1$!

Such sequences are called Uniformly integrable.

Definition 1 :- A collection of random variables $\{X_n\}_{n\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be Uniformly integrable if

$$\sup_{n\geq 1} E[|X_n| \mathbb{1}_{|X_n| > k}] \longrightarrow 0 \quad \text{as} \quad k \to \infty.$$

Theorem 3 (Uniform Integrability - Convergence)

If $\{X_n\}_{n\geq 1}$ are uniformly integrable then

(i) $\sup_{n\geq 1} E|X_n| < \infty$.

In addition, if $X_n \to X$ as $n \to \infty$ then

(ii) $E|X| < \infty$

(iii) $E[X_n] \longrightarrow E[X]$ as $n \to \infty$.

Proof :-

(i) let $\varepsilon = 1$ be given.

$\exists\, k_0 > 0$ s.t.

$$\sup_{n\geq 1} E[|X_n| \mathbb{1}_{|X_n| > k_0}] \leq 1 \quad -(5)$$

[As $\{X_n\}_{n\geq 1}$ are uniformly integrable]

Now, $$E[|X_n|] = E[|X_n| \mathbb{1}_{|X_n| \leq k_0}] + E[|X_n| \mathbb{1}_{|X_n| \geq k_0}]$$

$$\underset{(5)}{\leq} k_0 + 1 \qquad \forall n \geq 1$$

$$\Rightarrow \quad \sup_{n\geq 1} E[|X_n|] \leq k_0 + 1$$

(ii) $$E[|X|] = E\left[\liminf_{n\to\infty} |X_n|\right] \leq \liminf_{n\to\infty} E|X_n| < \infty \quad -(6)$$

For all $n \geq 1$, $$|E[X_n] - E[X]| \leq E[|X_n - X|] \quad -(7)$$

Let $\varepsilon > 0$ be given. For all $k \geq 1$,

$$E[|X_n - X|] = E[|X_n - X| \mathbb{1}_{|X_n - X| \leq k}] + E[|X_n - X| \mathbb{1}_{|X_n - X| > k}] \quad -(8)$$

We will estimate the 2nd term in (8)-RHS.

As $|X_n - X| \leq |X_n| + |X| \leq 2\max(|X_n|, |X|)$

$$|X_n - X| \mathbb{1}_{|X_n - X| > k}$$

$$\leq 2\left[\max(|X_n|, |X|) \mathbb{1}_{\max(|X_n|, |X|) > \frac{k}{2}}\right]$$

$$= 2\left[|X_n| \mathbb{1}_{(|X_n| > \frac{k}{2},\ |X_n| \geq |X|)} + |X| \mathbb{1}_{(|X| > \frac{k}{2},\ |X| > |X_n|)}\right]$$

$$\leq 2\left[|X_n| \mathbb{1}_{|X_n| > \frac{k}{2}} + |X| \mathbb{1}_{|X| > \frac{k}{2}}\right] \quad -(9)$$

By (5) and (6) $\exists\, k_1 > 0$ such that

$$\sup_{n \geq 1} E\left[|X_n| \mathbb{1}_{|X_n| > \frac{k_1}{2}}\right] < \varepsilon \quad \& \quad E\left[|X| \mathbb{1}_{|X| > \frac{k_1}{2}}\right] < \varepsilon \quad -(10)$$

$\overset{(9)}{\underset{(10)}{\Rightarrow}}$ $$\sup_{n \geq 1} E\left[|X_n - X| \mathbb{1}_{|X_n - X| > k_1}\right] < 2\varepsilon \quad -(11)$$

Take $K = K_1$ in in (8) to get, along with (11)

$$E[|X_n - X|] \le E[|X_n - X| \mathbb{1}_{|X_n - X| \le K_1}] + 2\varepsilon \quad -(12)$$
$$\forall n \ge 1$$

Finally
- $|X_n - X| \mathbb{1}_{|X_n - X| \le K_1} \longrightarrow 0$ as $n \to \infty$
- $|X_n - X| \mathbb{1}_{|X_n - X| \le K_1} \le K_1 \quad \forall n \ge 1$

$\therefore$ By Theorem 12.2 (Bounded Convergence Theorem)

$$E[|X_n - X| \mathbb{1}_{|X_n - X| \le K_1}] \longrightarrow 0 \text{ as } n \to \infty$$

$\therefore \exists N_1 > 1$ s.t.

$$E[|X_n - X| \mathbb{1}_{|X_n - X| \le K_1}] < \varepsilon \quad \forall n \ge N_1 \quad -(13)$$

Using (13) in (12) we have

$$E|X_n - X| \le 3\varepsilon \quad \forall n \ge N_1$$

Now using (7) yields the result $\square$

Remark:-

[a.s. Considerations]

- Assume $(\Omega, \mathcal{F}, \mathbb{P})$ to be complete.

  i.e. $A \in \mathcal{F}$ and $\mathbb{P}(A) = 0 \Rightarrow B \subseteq A$ then $B \in \mathcal{F}$.

$\Rightarrow Y : \Omega \to \mathbb{R} \quad \mathbb{P}(X = Y) = 1 \Rightarrow Y$ is a random variable

- also $E[X] = E[Y]$.

$\Rightarrow$ (Expected values do not change if we modify r.v's on set of measure 0)

· Conventions :- · $X = Y$ if $\mathbb{P}(X = Y) = 1$

- A property A holds a.s if $\mathbb{P}(A) = 1$ (almost surely)

· $X \geq 0$ a.s. $\Rightarrow$ $E[X] \geq 0$

· [Fatou's lemma] Suppose $\exists c > 0$ st $X_n \geq c$ a.s. $\forall n \geq 1$ then

$$E[\liminf_{n\to\infty} X_n] \leq \liminf_{n\to\infty} E[X_n]$$

· [Bounded / Dominated Convergence Theorem]

$X_n \to X$ a.s as $n \to \infty$ and

$|X_n| \leq Y$ a.s with $E[Y] < \infty$

then $E[X_n] \longrightarrow E[X]$

· [Monotone Convergence Theorem]

$\exists c$: $c \leq X_n$ and $X_n \leq X_{n+1}$ a.s. $\forall n \geq 1$

and $X_n \uparrow X$ as $n \to \infty$

Then $E[X_n] \uparrow E[X]$ as $n \to \infty$

Lemma 1: Suppose X on $(\Omega, \mathcal{F}, \mathbb{P})$ is an integrable r.v. Then $\forall \varepsilon > 0$ $\exists \delta > 0$ st. $A \in \mathcal{F}$ $\mathbb{P}(A) < \delta$

$$\Rightarrow \int_A |X| d\mathbb{P} \equiv E[|X| \mathbb{1}_A] < \varepsilon$$

Proof: let $\varepsilon > 0$ be given.

$E|X| < \infty$ & $E\left[|X| \mathbb{1}_{|X| < k}\right] \uparrow E|X|$ as $k \to \infty$ by M.C.T.

$$\Rightarrow E\left[|X| \mathbb{1}_{|X| > k}\right] = E\left[|X| \mathbb{1}_{|X| < k}\right] - E[|X|]$$

$$\longrightarrow 0 \quad \text{as } k \to \infty.$$

$\therefore$ let $k_1 > 0$ be such that

$$E\left[|X| \mathbb{1}_{|X_1| > k_1}\right] < \varepsilon$$

let $\delta = \frac{\varepsilon}{k_1}$ and $\mathbb{P}(A) < \delta$

$$E\left[|X| \mathbb{1}_A\right] = E\left[|X| \mathbb{1}_{|X_1| > k_1} \mathbb{1}_A\right] + E\left[|X| \mathbb{1}_{|X_1| \le k_1} \mathbb{1}_A\right]$$

$$\le E\left[|X| \mathbb{1}_{|X| > k_1}\right] + k_1 \mathbb{P}(A)$$

$$< \varepsilon + k_1 \delta = 2\varepsilon$$

As $\varepsilon > 0$ was arbitrary we are done. $\square$

lemma 2: $X \ge 0$ r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$. The following are equivalent:

ⓐ $X = 0$ a.s.

ⓑ $\int X \, d\mathbb{P} = 0$

Proof :- • Suppose $X = 0$ a.s.

S - simple $S \le X \Rightarrow$

$E_i \subseteq \{X \neq 0\}$

$S = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}$ then $\mathbb{P}(E_i) = 0$

$|\{a_1 .. a_n\}| = n$

$a_i \neq 0$

$\Rightarrow \quad E[S] = 0$

S - arbitrary $\Rightarrow \quad E[X] = 0.$

• let $X \geq 0$ and $E[X] = 0$

$$\{X \neq 0\} = \bigcup_{n=1}^{\infty} \{X > \tfrac{1}{n}\} \quad -(1)$$

Now

$$0 = \int x \, d\mathbb{P} \geq \int_{X > \frac{1}{n}} X \, d\mathbb{P} \geq \frac{1}{n} \mathbb{P}(X > \tfrac{1}{n})$$

$$\Rightarrow \quad \mathbb{P}(X > \tfrac{1}{n}) = 0. \quad -(2)$$

$$\overset{(1)}{\underset{(2)}{\Rightarrow}} \quad \mathbb{P}(X \neq 0) = 0 \quad (\text{Ex})$$

$$\Rightarrow \quad X = 0 \text{ a.s} \quad \square$$

16. Moment generating functions

Recall that for a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ the k^{th} moment of X is given by $E[X^k]$.

Also as $|X|^j \le |X|^k + 1 \quad \forall j \le k$

$\Rightarrow$ $E[X^k] < \infty$ then

$E[X^j] < \infty$ for $j \le k$.

Definition 1 :- The moment generating function of X $M_X(\cdot)$ is defined as

$$M_X(s) = E[e^{sX}] \quad \text{for } s \in \mathbb{R} \text{ for}$$

which this is finite. Note that $s \to e^{sX}$ is a non-negative r.v $\Rightarrow$ expectation always exist.

In analysis $M_X(\cdot)$ is called the Laplace transform of X.

Suppose $\exists\ s_0 \in \mathbb{R}_+$ s.t.

$M_X : (-s_0, s_0) \to \mathbb{R}$ is well defined

i.e. $E[e^{sX}] < \infty$ for $s \in (-s_0, s_0)$.

Now as $e^{|sX|} \leq e^{sX} + e^{-sX}$

$$\Rightarrow \quad E[e^{|sX|}] < \infty \quad \forall\ s \in (-s_0, s_0) \quad \text{— (1)}$$

Fact: $e : \mathbb{R}_+ \to [1, \infty)$

$$e^t := \sum_{k=0}^{\infty} \frac{t^k}{k!}$$ [Power series Expansion]

i.e. $\forall\ t \in \mathbb{R}_+$

$$T_n(t) = \sum_{k=1}^{n} \frac{t^k}{k!} \longrightarrow e^t \text{ as } n \to \infty$$

$(T_n(t) \geq 0 \ \& \ T_n(t) \uparrow e^t)$

let $s \in (-\delta_0, +\delta_0)$ be fixed

$$\Rightarrow |X|_n(s) = \sum_{k=1}^{n} \frac{|sX|^k}{k!} \geq 0$$

and $|X|_n(s) \uparrow e^{|sX|}$ as $n \to \infty$

Now; $$X_n(s) = \sum_{k=1}^{n} \frac{s^n X^n}{n!} \longrightarrow e^{sX} \text{ as } n \to \infty,$$

$$|X_n(s)| \leq |X|_n(s) \leq e^{|sX|} \quad \forall n \geq 1 \quad -(2)$$

and $$E[X_n(s)) = \sum_{k=1}^{n} \frac{s^n E[X^n]}{n!} < \infty \text{ by } (1), (2).$$

Using (1) and Dominated Convergence Theorem 15.2

$$E[X_n(s)] \longrightarrow E[e^{sX}] \text{ as } n \to \infty$$

$$\Rightarrow \quad M_X(s) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{E[X^k] s^k}{k!}$$

$$:= \sum_{k=1}^{\infty} \frac{E[X^k] s^k}{k!}$$

Fact :- $f : (-r, r) \to \mathbb{R}$ for some $r > 0$ and

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ is well defined}$$

then f is differentiable in $(-r, r)$ and

$$f'(x) = \sum_{k=1}^{\infty} k x^{k-1} a_k . \quad \forall x \in (-r, r) \quad -(3)$$

is well defined.

Repeated application of (3.) results in

f being k-times differentiable in $(-r,r)$

and
$$f^{(k)}(x) = \sum_{n=k}^{\infty} \prod_{j=1}^{k} (n-j+1)\, a_n x^{n-k}$$

Putting all of the above together we have

Theorem 1: Let $s_0 > 0$ and X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$M_X : (-s_0, s_0) \to \mathbb{R} \quad \text{given by}$$

$$M_X(s) = E[e^{sX}] \text{ is well defined.}$$

Then :-

- $E[X^n] < \infty \quad \forall\, n \geq 1$
- $$M_X(s) = \sum_{k=0}^{\infty} \frac{E[X^k]\, s^k}{k!}$$
- $$M_X^{(n)}(s) = \sum_{k=n}^{\infty} \prod_{j=1}^{n} (k-j+1) \frac{E[X^k]}{k!} s^{k-n}$$
- $$M_X^{(1)}(0) = E[X], \quad M_X^{(n)}(0) = E[X^n]$$

Example 2 :-

$$X \sim \text{Normal}(0,1)$$

$$Y = e^X$$

$$E[Y^n] = E[e^{nX}] \overset{\text{#8 HW4}}{=} \int_{-\infty}^{\infty} e^{ny} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy$$

$$= e^{\frac{n^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-n)^2}{2}}}{\sqrt{2\pi}}\, dy = e^{\frac{n^2}{2}} \quad \forall n \geq 1$$

$$\Rightarrow \quad M_X(t) = \infty \quad \forall\, t > 0$$

$$M_X(t) \leq 1 \quad \forall\, t \leq 0. \qquad \square$$

- Z be a random variable

$$\mathbb{P}(Z \in A) = \int_A \frac{e^{-\frac{(\log x)^2}{2}}}{\sqrt{2\pi}\, x} (1 + \sin(2\pi \log x))\, dx$$

for $A \in \mathcal{B}_{\mathbb{R}}$

Ex: $\quad E[Z^n] = e^{n^2/2}. \quad \forall n \geq 1$

& $Z \neq Y$ (in distribution)

Theorem 2:- Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$

(a) If X and Y are bounded then

$X \stackrel{d}{=} Y$ (i.e. $\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x)$ $\forall x \in \mathbb{R}$)

$$\Updownarrow$$

$$E[X^k] = E[Y^k] \quad \forall k \geq 1$$

(b) Suppose $M_X(t) = M_Y(t) \in \mathbb{R}$ for $t \in (-\delta, \delta) \Rightarrow X \stackrel{d}{=} Y$