22 Strong law of large Numbers

Example 1 : Perform an experiment n times & we are interested in an event A. We observe,

$$X_i = \begin{cases} 1 & \text{if } A \text{ occurs in } i^{th} \text{ trial} \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \equiv \text{Relative frequency of } A$$

What is limiting relative frequency ?

Example 2 : Suppose $a_i \in \{0,1\}$ $\forall i \geq 1$.

$$\bar{a}_n = \frac{1}{n}\sum_{i=1}^{n} a_i$$

Does $\lim_{n\to\infty} \bar{a}_n$ exists ?

Example 3 :- Suppose p is the proportion of objects with characteristic α, in a box of N objects. We sample 'n' objects and set

$$X_i = \begin{cases} 1 & \text{if object } i \text{ has characteristic } \alpha \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \forall n \geq 1$$

Q: How close is $\bar{X}_n$ to p?

Theorem 1 :- let $X, \{X_n\}_{n\geq 1}$ i.i.d r.v on $(\Omega, \mathcal{F}, \mathbb{P})$.
Let $E|X| < \infty$. Then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{a.s.} E[X] \qquad \text{as } n \to \infty$$

Proof of Theorem 1 - (Special Case) $X \stackrel{d}{=}$ Bernoulli(p)

Let $S_n = \sum_{i=1}^{n} X_i \qquad n \geq 1.$

$$\bar{S} = \limsup_{n\to\infty} \frac{S_n}{n}, \quad \text{and} \quad \underline{S} = \liminf_{n\to\infty} \frac{S_n}{n}$$

- $0 \leq \bar{S}, \underline{S}, S_n \leq 1 \quad \forall n \geq 1.$

Claim 1: Show $E[\bar{S}] \leq E[X]$ — (1)

let

$$\tilde{X}_k = 1 - X_k, \quad \tilde{X} = 1 - X, \quad \tilde{S}_n = \frac{\sum_{i=1}^{n} \tilde{X}_i}{n}, \quad \bar{\tilde{S}} = \overline{\lim_{n\to\infty}} \frac{\tilde{S}_n}{n}$$

Assuming claim 1, and applying to $\{\tilde{X}_k\}_{k\geq 1}$, $\tilde{X}$

we have

$$E[\tilde{\bar{S}}] \leq E[\tilde{X}]$$

$$\text{(Ex)} \implies E[\underline{S}] \geq E[X] \quad -(2)$$

As

$$\underline{S} \leq \bar{S}, \quad -(3)$$

we have from (1), (2), and (3)

$$\bar{S} = \underline{S} \quad a.s.$$

$\therefore$ $\exists$ a r.v. S such that

$$\frac{S_n}{n} \xrightarrow{a.s.} S, \quad \text{as } n \to \infty \quad -(4)$$

Also For, $\delta > 0$,

$$\mathbb{P}\left(\left| \frac{S_n}{n} - E[X] \right| > \delta \right) \overset{\text{(Ex.)}}{\leq} \frac{\mathrm{Var}\left(\frac{S_n}{n}\right)}{\delta^2}$$

$$= \frac{1}{n^2} \frac{p(1-p)}{\delta^2}$$

$\therefore \quad \frac{S_n}{n} \xrightarrow{p} E[X] \quad$ as $n \to \infty$. — (5)

(4), (5) $\Rightarrow \quad \frac{S_n}{n} \xrightarrow{a.s} E[X] \quad$ as $n \to \infty$

$\therefore$ all we have to do is prove claim 1.

Proof of claim 1 :

let $\varepsilon > 0$ be given. for $k \geq 1$

$$N_k = \inf \left\{ n \in \mathbb{N} \;\middle|\; \frac{X_k + \cdots + X_{k+n-1}}{n} \geq \bar{s} - \varepsilon \right\}$$

Ex: $\cdot \{N_k\}_{k \geq 1}$ i.i.d. – r.v. and $N_k < \infty$ a.s.

$\cdot \; \exists \; m > 0 \qquad \mathbb{P}(N_k > m) < \varepsilon$ — (6)

let $Y_k = \begin{cases} X_k & \text{if } N_k \leq m \\ 1 & \text{if } N_k > m. \end{cases}$

and

$$N_k^Y = \inf \left\{ n \in \mathbb{N} \;\middle|\; \frac{Y_k + \cdots + Y_{k+n-1}}{n} \geq \bar{s} - \varepsilon \right\}$$

Note that $N_k^Y \leq N_k$ a.s.

and if $N_k \geq m \underset{\text{Ex.}}{\Rightarrow} N_k^Y = 1$

$$\Rightarrow N_k^Y \leq m \quad \text{a.s.}$$

$\therefore \ \forall \ n \geq m$

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{m} Y_i + \sum_{i=m+1}^{2m} Y_i + \cdots + \sum_{i=\cdots}^{n} Y_i$$

$$\geq \underbrace{(\bar{S}-\varepsilon) + (\bar{S}-\varepsilon) + \cdots +}_{n-m} 0$$

$$= (n-m)(\bar{S}-\varepsilon)$$

$$\Rightarrow \sum_{i=1}^{n} E[Y_i] \geq (n-m)(E(\bar{S})-\varepsilon)$$

$\Rightarrow$

$$\sum_{i=1}^{n} E(X_i \mathbb{1}_{N_i \leq m}) + P(N_i > m) \geq (n-m)(E(\bar{S})-\varepsilon)$$

As $E[X] = E[X_i] \geq E[X_i \mathbb{1}_{N_i \leq m}]$ and using 6

The above $\Rightarrow$

$$n\,(E[X]+\varepsilon) \geq (n-m)(E[\bar{S}]-\varepsilon)$$

$$\Rightarrow \quad E[\bar{X}]+\varepsilon \geq \left(1-\frac{m}{n}\right)(E[\bar{S}]-\varepsilon)$$

$$\forall\, n \geq m.$$

As $\varepsilon > 0$ was arbitrary $\Rightarrow$ (Ex.)

$$E[\bar{S}] \leq E[X]$$

$\square$

Proof of Theorem 1 – [following framework of special case proof]

We will prove the result in 3-steps.

Let $\quad S_n = \sum_{i=1}^{n} X_i \qquad n \geq 1.$

$$\bar{S} = \limsup_{n\to\infty} \frac{S_n}{n}, \quad \text{and} \quad \underline{S} = \liminf_{n\to\infty} \frac{S_n}{n}$$

<u>claim 2.:</u> Show $E[\bar{S}] \leq E[X]$ – (7)

Assuming claim 2., let

$$\tilde{X}_k = -X_k, \quad \tilde{X} = -X, \quad \tilde{S}_n = \frac{\sum_{i=1} \tilde{X}_i}{n}, \quad \bar{\tilde{S}} = \overline{\lim_{n\to\infty}} \frac{\tilde{S}_n}{n}$$

Using (7) we have

$$E[\bar{\tilde{S}}] \leq E[\tilde{X}]$$

(Ex) $\Rightarrow$ $E[\underline{S}] \geq E[X]$ — (8)

As

$$\underline{S} \leq \bar{S}, \quad \text{— (9)}$$

we have from (7), (8), and (9)

$$\bar{S} = \underline{S} \quad a.s.$$

$\therefore$ $\exists$ a r.v. S such that

$$\frac{S_n}{n} \xrightarrow{a.s.} S, \quad \text{as } n\to\infty \quad \text{— (10)}$$

Claim 3 :- S is measurable w.r.t tail σ-algebra

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma\{X_k : k \geq n\}$$

Assuming claim 3 by Kolmogorov 0-1 law $S = c$ a.s. for some constant c.

$$E[X] \leq E[\underline{S}] = E[S]$$

$$E[X] \geq E[\bar{S}] = E[S]$$

$$\Rightarrow \quad c = E[X] \qquad \square$$

Proof of claim 2 : Fix $\alpha, \beta, \varepsilon > 0$.

Let $X^\beta = \max\{X, -\beta\}$, $X_k^\beta = \max\{X_k, -\beta\}$, $\bar{S}^\alpha = \max\{S, \alpha\}$

For $k \geq 1$,

$$N_k = \inf\left\{ n \in \mathbb{N} \;\middle|\; \frac{X_k^\beta + \cdots + X_{k+n-1}^\beta}{n} \geq \bar{S}^\alpha - \varepsilon \right\}$$

Ex: • $\{N_k\}_{k \geq 1}$ i.i.d. r.v. and $N_k < \infty$ a.s.

• $\exists\, m > 0$ $\quad \mathbb{P}(N_k > m) < \varepsilon$ $\quad$ - (11)

let $Y_k = \begin{cases} X_k^\beta & \text{if } N_k \le m \\ \max(\alpha, X_k^\beta) & \text{if } N_k > m. \end{cases}$

and

$$N_k^Y = \inf\left\{ n \in \mathbb{N} \;\middle|\; \frac{Y_k + \cdots + Y_{k+n-1}}{n} \geq \bar{S}^\alpha - \varepsilon \right\}$$

Now $Y_k \geq X_k^\beta \Rightarrow N_k^Y \le N_k$

If $k \geq 1$ & $N_k > m$

$\Rightarrow Y_k = \max(\alpha, X_k^\beta) \geq \alpha \geq \bar{S}^\alpha \geq \bar{S}^\alpha - \varepsilon$

$\Rightarrow N_k^Y = 1$

$\Rightarrow N_k^Y \le m$ a.s.

$\therefore \forall n \geq m$

$$\sum_{i=1}^n Y_i = \sum_{i=1}^m Y_i + \sum_{i=m+1}^{2m} Y_i + \cdots + \sum_{i=\cdots}^{n} Y_i$$

$$\geq \underbrace{(\bar{S}^\alpha - \varepsilon) + (\bar{S}^\alpha - \varepsilon) + \cdots +}_{n-m} m(-\beta)$$

$$= (n-m)(\bar{S}^\alpha - \varepsilon) + m(-\beta)$$

$$\Rightarrow \sum_{i=1}^n E[Y_i] \geq (n-m)\left(E(\bar{S}^\alpha) - \varepsilon\right) - m\beta$$

$\Rightarrow$

$$\sum_{i=1}^{n} E(X_i^{\beta} 1_{N_i \le m}) + E[\min(\alpha, X^{\beta}) 1_{N_i > m}]$$
$$\ge (n-m)(E(\bar{S}^{\alpha}) - \varepsilon) - m\beta$$

$\Rightarrow$ (Ex)

$$\sum_{i=1}^{n}\left(E[X_i^{\beta}] + (\alpha+\beta)\,\mathbb{P}(N_i > m)\right)$$
$$\ge (n-m)(E(\bar{S}^{\alpha}) - \varepsilon) - m\beta$$

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$$\Rightarrow \quad n\,EX^{\beta} + (\alpha+\beta)\varepsilon \ge (n-m)(E(\bar{S}^{\alpha}) - \varepsilon) - m\beta$$

Dividing by n, let $n \to \infty$ and as $\varepsilon > 0$ was arbitrary

$$\Rightarrow \quad E[\bar{S}^{\alpha}] \le E[X^{\beta}] \quad - (*)$$

Now • $|X^{\beta}| \le |X|$; $X^{\beta} \longrightarrow X$ as $\beta \to \infty$

[Take along any sequence $\beta_n \to \infty$]

and $E|X| < \infty$

$$\Rightarrow \quad E[X^{\beta}] \longrightarrow E[X] \quad \text{as } \beta \to \infty$$

$-$ (12)

• $$\bar{S}^{\alpha} 1_{(S \ge 0)} \uparrow \bar{S} 1_{(S \ge 0)}$$

$\therefore$ M.C.T.

$$\Rightarrow \quad E(\bar{S}^{\alpha} 1_{(S \ge 0)}) \uparrow E[\bar{S} 1_{(S \ge 0)}] \quad \text{as } \alpha \to \infty \quad - (13)$$

[Take along any sequence $\alpha_n \to \infty$]

- $$-\bar{S} \le \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} |X_i|}{n}$$

$$\Rightarrow \quad -E[\bar{S}] \overset{\text{(Fatou)}}{\le} \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} E|X_i|}{n} = E|X_1|$$

$$\Rightarrow \quad E[\bar{S}] \ge -E|X_1| > -\infty$$

$$\therefore \quad E[\bar{S}^{\alpha} 1_{\bar{S}<0}] = E[\bar{S} 1_{\bar{S}<0}] < \infty. \quad \text{— (14)}$$

(13) and (14)

$\Rightarrow$

$$E[\bar{S}^{\alpha}] = E[\bar{S}^{\alpha} 1_{\bar{S}\ge 0}] + E[\bar{S}^{\alpha} 1_{\bar{S}<0}]$$

$$\longrightarrow E[\bar{S} 1_{\bar{S}\ge 0}] + E[\bar{S} 1_{\bar{S}<0}]$$

as $\alpha \to \infty$.

$$= E[\bar{S}]. \quad \text{— (15)}$$

$\therefore$ Taking limits in (*) using (12) and (15)

$$\Rightarrow \quad E[\bar{S}] \le E[X].$$

$\square$

Proof of claim 3 (b): let $k \geq 1$,

$$A = \left\{ \lim_{n\to\infty} \frac{S_n}{n} \geq \alpha \right\}, \quad \text{and}$$

$$B = \bigcap_{m=k}^{\infty} \left\{ \omega : \lim_{n\to\infty} \frac{\sum_{i=m}^{n} X_i}{n} \geq \alpha \right\}.$$

It is enough to show

$$A = B \quad — (18)$$

[Ex: (18) $\Rightarrow$ S is m'ble w.r.t. τ]

First note,

$$A = \left\{ \omega \in \Omega \,\middle|\, \forall \varepsilon > 0 \ \exists N(\omega) : \frac{S_n}{n} \geq \alpha - \varepsilon \ \forall n \geq N \right\}$$

and

$$B = \left\{ \omega \in \Omega \,\middle|\, \forall m \geq k, \forall \varepsilon > 0 \ \exists N(\omega) : \frac{\sum_{i=m}^{n} X_i}{n} \geq \alpha - \varepsilon \ \forall n \geq N \right\}$$

Take $\omega \in A$. let $\varepsilon > 0$, $m \geq k$ be given

$\exists N(\omega)$ such that

$$\frac{|X_1| + \cdots + |X_m|}{n} \leq \frac{\varepsilon}{2} \qquad \forall n \geq N$$

$\Rightarrow$ $\exists N_1(\omega)$ $\forall n \geq N_1(\omega)$

$$\frac{S_n(\omega)}{n} \geq \alpha - \frac{\varepsilon}{2} \qquad \forall n \geq N_1$$

$N_2 = \max\{N, N_1\}$

$$\frac{\sum_{i=m}^{n} X_i}{n} = \frac{S_n}{n} - \frac{\sum_{i=1}^{m} X_i}{n}$$

$$\geq \alpha - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \qquad \forall\, n \geq N$$

$$\geq \alpha - \varepsilon$$

$$\Rightarrow \quad \omega \in B.$$

Take $\omega \in B$

let $\varepsilon > 0$ be given.

Take $m = k$ $\Rightarrow$

$$\exists N_3: \quad \frac{S_n}{n} - \frac{\sum_{i=1}^{n} X_k}{n} \geq \alpha - \frac{\varepsilon}{2} \qquad \forall\, n \geq N_3$$

$$\exists N_4: \quad \frac{\sum_{i=1}^{n} |X_k|}{n} \leq \varepsilon/2 \qquad \forall\, n \geq N_4$$

So $n \geq N_5 = \max(N_3, N_4)$

$$\frac{S_n}{n} \geq \alpha - \varepsilon$$

$$\Rightarrow \quad \omega \in A. \qquad \square$$