Recall :-

- let μ be a Probability measure on $(\mathbb{R}, \mathcal{B}_{(\mathbb{R})})$ with characteristic function

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{itx} \mu(dx)$$

Then if $a<b$ $\quad \frac{1}{2}\mu(\{a\}) + \frac{1}{2}\mu(\{b\}) + \mu(a,b) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \hat{\mu}(t)\, dt$

- let X, Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$\phi_X(t) = \phi_Y(t) \quad \forall\ t \in \mathbb{R}$$

$$\Rightarrow \quad \mathbb{P}\circ X^{-1}(\cdot) = \mathbb{P}\circ Y^{-1}(\cdot)$$

Before moving to the second objective, we will connect weak-convergence with other modes discussed. A-priori $X_n \xrightarrow{d} X$, then they could all be on different probability spaces. There is still one equivalence that can be formulated.

Proposition 1 (Skorokhod): let $\{X_n\}_{n\geq 1}$ and X be random variables with distribution functions $\{F_n\}_{n\geq 1}$ and F respectively.

The following are equivalent:-

(a) $F_n \longrightarrow F$ pointwise at all continuity points of F, as $n\to\infty$.

(b) $\exists\ (\Omega, \mathcal{F}, \mathbb{P})$ – a Probability space

& r.v.'s $\{Y_k\}_{k\geq 1}$, Y on it such that

$$Y_k \xrightarrow{a.e.} Y \quad \text{as } k \to \infty \quad \text{and}$$

$\forall k \geq 1$, $Y_k \overset{d}{=} X_k$ and $Y \overset{d}{=} X$.

Proof :- (b) $\Rightarrow$ (a)

$$Y_k \xrightarrow{a.e.} Y$$

$g : \mathbb{R} \to \mathbb{R}$ & bounded continuous function

$$\Rightarrow \quad g(Y_k) \xrightarrow{a.e.} g(Y) \qquad \text{as } k \to \infty$$

$$\overset{D.C.T.}{\Longrightarrow} \quad E[g(Y_k)] \longrightarrow E[g(Y)] \quad \text{as } k \to \infty$$

$$\overset{Y_k \overset{d}{=} X_k}{\underset{Y \overset{d}{=} X}{\Rightarrow}} \quad \tilde{E}_k[g(X_k)] \longrightarrow \tilde{E}[g(X)] \quad \text{as } k \to \infty$$

$$\Rightarrow \quad \tilde{\mathbb{P}}_k \cdot X_k^{-1} \overset{w}{\Longrightarrow} \tilde{\mathbb{P}} \cdot X \quad \text{as } k \to \infty$$

$\Rightarrow$ Theorem 1 (v) implies (a) $\square$

(a) $\Rightarrow$ (b)

let $\quad F_n(x) = \tilde{\mathbb{P}}_n(X_n \leq x) \quad \forall\, x \in \mathbb{R}$

$$F(x) = \tilde{\mathbb{P}}(X \leq x)$$

Take $\Omega = [0,1]$, $\mathcal{B}_{[0,1]}$, $\mathbb{P}(d\omega) \equiv$ Uniform

$Y_n : \Omega \to \mathbb{R}$; $Y_n(\omega) = \inf\{x \in \mathbb{R} : \omega \le F_n(x)\}$

and

$Y : \Omega \to \mathbb{R}$; $Y(\omega) = \inf\{x \in \mathbb{R} : \omega \le F(x)\}$

$$\Rightarrow \quad \tilde{\mathbb{P}}_n \cdot X_n^{-1} = \mathbb{P} \cdot Y_n^{-1} \qquad \left(\text{i.e } Y_n \overset{d}{=} X_n \;\; n \ge 1 \right)$$

$$\tilde{\mathbb{P}} \cdot X^{-1} = \mathbb{P} \cdot Y^{-1} \qquad Y \overset{d}{=} X$$

$F_n \longrightarrow F$ as $n \to \infty$ at all continuity points of F.

- $\omega \in \Omega$, let $\varepsilon > 0$ be s.t. $\alpha := Y(\omega) - \varepsilon$ is a continuity point of F

$\Rightarrow \quad Y(\omega) > \alpha \quad \Rightarrow \quad F(\alpha) < \omega$

$F_n(\alpha) \to F(\alpha)$

$\Rightarrow \quad \exists m$ s.t. $F_n(\alpha) < \omega \quad \forall n \ge m$

$\Rightarrow \quad \exists m$ s.t $Y_n(\omega) > \alpha \quad \forall n \ge m$

$$\Rightarrow \quad \liminf_{n \to \infty} Y_n(\omega) \ge \alpha = Y(\omega) - \varepsilon$$

$\overset{Ex}{\Longrightarrow}$ $\liminf_{n\to\infty} Y_n(\omega) \geq Y(\omega)$.

Working similarly (Ex.) $\limsup_{n\to\infty} Y_n(\omega) \leq Y(\omega)$, $\forall\ \omega \in \Omega$.

$\left(\text{Y has countably many discontinuities}\right) \Rightarrow Y_n(\omega) \longrightarrow Y(\omega)$ a.e. $\mathbb{P}$.

$\square$

Corollary 2 : Suppose $\{X_n\}_{n\geq 1}$ and X are random variables with distribution functions $\{F_n\}_{n\geq 1}$ and F respectively. Then the following are equivalent

(a) $F_n \longrightarrow F$ pointwise for all continuity points of F

(b) $X_n \xrightarrow{w} X$ as $n\to\infty$

Proof:-

(b) $\Rightarrow$ (a) follows from Theorem 1 (v).

(a) $\Rightarrow$ (b)

By Proposition 1 $\exists$ a Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

& r.v's $\{Y_k\}_{k\geq 1}$, Y on it such that

$$Y_k \xrightarrow{a.e.} Y \quad \text{as } k\to\infty \quad \text{and}$$

$\forall k\geq 1$, $Y_k \stackrel{d}{=} X_k$ and $Y \stackrel{d}{=} X$.

let $g:\mathbb{R}\to\mathbb{R}$ a bounded continuous function

$$\Rightarrow \quad g(Y_k) \xrightarrow{a.e.} g(Y) \quad \text{as } k\to\infty$$

$$\overset{D.C.T.}{\Longrightarrow} \quad E[g(Y_k)] \longrightarrow E[g(Y)] \quad \text{as } k\to\infty$$

$$\underset{Y \stackrel{d}{=} X}{\overset{Y_k \stackrel{d}{=} X_k}{\Longrightarrow}} \quad \tilde{E}_k[g(X_k)] \longrightarrow \tilde{E}[g(X)] \quad \text{as } k\to\infty$$

$$\Rightarrow \quad \tilde{\mathbb{P}}_k\cdot X_k^{-1} \overset{w}{\Longrightarrow} \tilde{\mathbb{P}}\cdot X. \quad \text{as } k\to\infty$$

$\Rightarrow$ Theorem 1 (v) implies ⓐ $\square$

Theorem 2 (Helly Selection Theorem)

$n\geq 1$, $f_n:\mathbb{R}\to\mathbb{R}$ - increasing functions & uniformly bounded.

Then $\exists \{f_{n_k}\}_{k\geq 1}$ - a subsequence that converges pointwise on $\mathbb{R}$.

Proof: Ex. Page 167 - Rudin - PMA $\square$

Remark 2:

- $\{F_n\}_{n\geq 1}$ are a sequence of distribution functions. Then Theorem 2 $\Rightarrow$

$\exists F_{n_k}$ such that $F_{n_k} \longrightarrow F$ pointwise.

- $F(x) \le F(y)$ if $x \le y$ ✓
- $0 \le F(x) \le 1 \quad \forall x \in \mathbb{R}$ ✓

(only at continuity point) $\star_1$ - F - right continuous (can construct)

$\star_2$ $\boxed{F(-\infty) > 0 \text{ and } F(\infty) < 1}$

We need extra condition to avoid $\star_2$

E.g:- $X_n \sim N(n,1)$, $\mu_n^c = \delta_n(\cdot)$

Definition 2 : We say a collection of measures $\{\mu_n\}_{n\ge1}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ are tight if

$\forall \varepsilon > 0 \quad \exists \; a < b$ s.t. $\mu_n([a,b]) > 1-\varepsilon \quad \forall n \ge 1$

- no escape of mass to "infinity". - relative compactness condition

Theorem 3 :- Let $\{\mu_n\}_{n\ge1}$ be a tight sequence of probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then $\exists$ a subsequence $\{\mu_{n_k}\}_{k\ge1}$ and a probability measure μ such that.

$$\mu_{n_k} \Rightarrow \mu.$$

Proof :- By Theorem 2 and Remark 2

let $F_n(x) = \mu_n((-\infty, x])$

$\exists \{F_{n_k}\}_{k\geq 1}$ such that

$F_{n_k} \longrightarrow F$ at all continuity points of F ; $0 \leq F(x) \leq 1$

$\{\mu_n\}_{n\geq 1}$ are tight $\Rightarrow \forall \varepsilon > 0$; $\exists$ $a < b$

$\mu_n((a,b]) \geq 1-\varepsilon \quad \forall n \geq 1$

$F(b) - F(a) \geq 1-\varepsilon$

$\Rightarrow \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x) \geq 1-\varepsilon$

$\Rightarrow \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x) = 1$

$\Rightarrow \lim_{x\to\infty} F(x) = 1$ & $\lim_{x\to-\infty} F(x) = 0$.

$\therefore$ $F_{n_k} \longrightarrow F$ at all continuity points of F and F is a distribution function.

let μ be a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

be given by $\mu((a,b]) = F(b) - F(a)$

By Corollary 1:

$$\mu_{n_k} \Longrightarrow \mu \quad \text{as } k \to \infty \quad \square$$

Proposition 2 [Tightness vs characteristic functions]

Let $\{\mu_n\}_{n\geq 1}$ be a sequence of Probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with characteristic functions $\phi_n(\cdot)$, respectively. Suppose:

- $\exists\, \delta > 0$: $\phi_n(t) \longrightarrow g(t)$ as $n \to \infty$ $\forall\, |t| < \delta$
- $g(\cdot)$ is continuous at 0.

Then $\{\mu_n\}_{n\geq 1}$ is tight.

Proof:– Let $\varepsilon > 0$ be given.

Now, $\phi_n(0) = 1 \;\; \forall\, n \geq 1 \implies g(0) = 1$

By hypothesis:

(Ex.) $\therefore\ \exists\, \eta > 0$ and N:

$$|1 - \phi_n(t)| \leq \varepsilon \qquad \forall\, |t| < \eta, \;\; \forall\, n \geq N.$$

$\text{—} \bigcirc$

Idea: Find $\beta > 0$ to relate $\mu_n(|x| > \beta)$ with $\frac{1}{\eta} \int_{-\eta}^{\eta} (1 - \varphi_n(t))\,dt$

let $\eta > 0$, $n \geq N$.

$$\int_{-\eta}^{\eta} (1 - \varphi_n(t))\,dt$$

$$= \int_{-\eta}^{\eta} \left(\int_{\mathbb{R}} (1 - e^{itx})\, d\mu_n(x) \right) dt$$

(Fubini Ex.)
$$= \int_{\mathbb{R}} \int_{-\eta}^{\eta} (1 - e^{itx})\, dt\, d\mu_n(x)$$

$$= \int_{\mathbb{R}} \int_{-\eta}^{\eta} (1 - \cos(tx))\, dt\, d\mu_n(x)$$

$$= \int_{\mathbb{R}} 2\left(\eta - \frac{\sin(\eta x)}{x}\right) d\mu_n(x) \quad -(1)$$

- $2 - \frac{\sin(\eta x)}{x} \geq 0$ & $\quad -(2)$

- $|x| \geq \frac{2}{\eta} \Rightarrow$

$$2\left(\eta - \frac{\sin \eta x}{x}\right) = 2\eta\left(1 - \frac{\sin \eta x}{\eta x}\right) \geq 2\eta\left(1 - \frac{1}{\eta |x|}\right) \geq \eta \quad -(3)$$

$\therefore$ ① ② $\Rightarrow$

$$\int_{-\eta}^{\eta} (1-\varphi_n(t)) \geq \int_{|x|>\frac{2}{\eta}} 2\left(\eta - \frac{\sin(\eta x)}{x}\right) d\mu_n(x)$$

$$\geq \eta \, \mu_n\left(|x| > \frac{2}{\eta}\right) \quad -④$$

$\therefore$ For $n \geq N$ from ⓪ and ④

$\exists \eta > 0$

$$\mu_n\left(|x| > \frac{2}{\eta}\right) \underset{④}{\leq} \frac{1}{\eta} \int_{-\eta}^{\eta} (1-\varphi_n(t))\, dt$$

$$\underset{⓪}{<} \varepsilon$$

$\exists$ $\beta_1, \ldots, \beta_N$ such that

$$\mu_k(|x| > \beta_k) < \varepsilon \quad \forall\, k = 1, \ldots N$$

let $K = \max\left(\beta_1, \beta_2, \ldots, \beta_N, \frac{2}{\eta}\right)$

$\Rightarrow \quad \mu_n(|x| > k) < \varepsilon \quad \forall n \geq 1$

$\therefore \quad \mu_n([-k, k]) \geq 1 - \varepsilon \quad \forall n \geq 1$

$\Rightarrow \quad \{\mu_n\}_{n \geq 1}$ are tight.

$\square$

We are now read to resolve the final objective.

Theorem 4 (Continuity) :

let $\{\mu_n\}_{n\geq 1}$ be a sequence of Probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with characteristic functions $\phi_n(\cdot)$, respectively. Then

$$\mu_n \Rightarrow \mu \quad \Longleftrightarrow \quad \phi_n(t) \longrightarrow \phi(t) \quad \forall t \in \mathbb{R}.$$

Proof:

$\Rightarrow$ let $\mu_n \Rightarrow \mu$. Then

$\forall t > 0$: $f(x) = \cos(tx)$ and $g(x) = \sin(tx)$ are bounded continuous functions on $\mathbb{R}$.

$$\Rightarrow \quad \int f(x)\, d\mu_n(x) \longrightarrow \int f(x)\, d\mu(x) \quad \text{as } n \to \infty,$$

$$\int g(x)\, d\mu_n(x) \longrightarrow \int g(x)\, d\mu(x) \quad \text{as } n \to \infty.$$

$$\therefore \forall t > 0, \quad \phi_n(t) = \int f(x)\, d\mu_n(x) + i \int g(x)\, d\mu_n(x)$$

$$\longrightarrow \phi(t) \quad \text{as } n \to \infty.$$

$\Leftarrow$ Suppose $\phi_n(t) \xrightarrow[\text{as } n \to \infty]{} \phi(t)$; $\forall t > 0$.

Then as $\phi(\cdot)$ is continuous at 0 (Ex.)

by Proposition 2

$\{\mu_n\}_{n \geq 1}$ is tight.

(Ex.) $\Rightarrow$ $\{\mu_{n_k}\}_{k \geq 1}$ is tight for all

subsequences as well.

Theorem 3 $\Rightarrow$ For every subsequence $\{\mu_{n_k}\}_{k \geq 1}$

$\exists$ a further subsequence $\{\mu_{n_{k_l}}\}_{l \geq 1}$ such that $\mu_{n_{k_\ell}} \Rightarrow \nu$ for some Probability measure.

From $\Rightarrow$) proof: we have

$$\phi_{n_{k_\ell}}(t) \longrightarrow \hat{\nu}(t) \quad \forall\, t > 0$$

as $\ell \to \infty$

But $\phi_{n_{k_\ell}}(t) \longrightarrow \phi(t) \quad \forall\, t > 0$ by

as $\ell \to \infty$

assumption.

$\therefore$ By Corollary 1: $\nu = \mu$.

So we have shown: For every subsequence $\{\mu_{n_k}\}_{k \geq 1}$

$\exists$ a further subsequence $\{\mu_{n_{k_\ell}}\}_{\ell \geq 1}$ such that

$$\mu_{n_{k_\ell}} \Rightarrow \mu \quad \text{as } \ell \to \infty$$

(Ex:) $\dashv$ $\mu_n \Rightarrow \mu$ as $n \to \infty$.

$\square$

21 Central limit Theorem :

Theorem 1 : let $\{X_k\}_{k\geq 1}$ be i.i.d. with finite mean μ and variance σ^2.

let

$$S_n = X_1 + X_2 + \cdots + X_n \quad \text{and} \quad Y_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

Then

$$\mathbb{P}\circ Y_n^{-1} \implies \mu_Z \quad \text{where}$$

$$\mu_Z(A) = \int_A \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz \quad \forall\ A \in \mathcal{B}_{\mathbb{R}}.$$

Proof :- Let $W_i = \dfrac{X_i - \mu}{\sigma} \quad \forall i \geq 1$

$\Rightarrow \{W_i\}_{i\geq 1}$ are i.i.d.

$$E[W_i] = 0, \quad Var[W_i] = 1$$

$$\& \quad Y_n = \frac{\sum_{i=1}^{n} W_i}{\sqrt{n}}$$

let $t > 0$.

$$\phi_{Y_n}(t) = E\left[e^{itY_n}\right] = E\left[e^{it\frac{\sum_{k=1}^{n} W_k}{\sqrt{n}}}\right]$$

$$\overset{\{W_i\}_{i\geq 1}}{\underset{\text{i.i.d.}}{=}} \left[E e^{\frac{itW_1}{\sqrt{n}}}\right]^n$$

Ex: • $\forall a \in \mathbb{R}$:

$$\left| e^{ia} - \left(1 + ia - \frac{a^2}{2}\right) \right| \leq \min\left(\frac{|a|^3}{6}, a^2 \right)$$

$$\therefore \quad \left| E\left[e^{\frac{itW_1}{\sqrt{n}}}\right] - 1 - \frac{t^2}{2n} \right|$$

$$\leq E\left| e^{\frac{itW_1}{\sqrt{n}}} - 1 - \frac{itW_1}{n} - \frac{t^2 W_1^2}{2n} \right|$$

$$\leq \frac{t^2}{2n} E\left[\min\left\{ \frac{|W_1|^3}{3\sqrt{n}}, W_1^2 \right\}\right]$$

let $d_n = E\left[\min\left\{ \frac{|W_1|^3}{3\sqrt{n}}, W_1^2 \right\}\right]$.

Ex: • $d_n \to 0$ as $n \to \infty$

$\Rightarrow$

$$E\left[e^{\frac{itW_1}{\sqrt{n}}}\right] = 1 - \frac{t^2}{2n} - \frac{h_n t^2}{2n} \qquad -(1)$$

with $h_n \in \mathbb{C}$ and $|h_n| \to 0$ as $n \to \infty$

Ex

- $z_n \to z$ as $n \to \infty$ in $\mathbb{C}$ $\Rightarrow$ $\left(1 + \frac{z}{n}\right)^n \longrightarrow e^z$ as $n \to \infty$

$\Rightarrow$

$$\phi_{Y_n}(t) = \left(1 - \frac{t^2}{2n} - \frac{t^2}{2n} h_n\right)^n$$

$$= \left(1 + \frac{-t^2/2\,(1-h_n)}{n}\right)^n$$

$$\longrightarrow e^{-t^2/2}$$

Ex: $\hat{\mu}_Z(t) = e^{-t^2/2} \quad \forall\, t \in \mathbb{R}$

$\therefore$ $(\mathbb{P}.Y_n^{-1})^\wedge(t) \longrightarrow \hat{\mu}_Z(t) \quad \forall\, t \in \mathbb{R}$ as $n \to \infty$

$\Rightarrow$ Theorem 20.4 $\quad \mathbb{P}.Y_n^{-1} \Longrightarrow \mu_Z$

$\square$