Recall Let X_n, X be a sequence of r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$.

- $X_n \xrightarrow{P} X$ if

  $\forall \varepsilon > 0$ $\quad \mathbb{P}(|X_n - X| > \varepsilon) \longrightarrow 0$ as $n \to \infty$.

- $X_n \xrightarrow{a.s} X$ if $\mathbb{P}(C) = 1$ where

$$C = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{ |X_n - X| < \tfrac{1}{k} \}$$

$$X_n \xrightarrow{a.s} X \quad \underset{\not\Leftarrow}{\Longrightarrow} \quad X_n \xrightarrow{P} X$$

(Borel – Cantelli)

- If $\{A_n : n \geq 1\}$ is a sequence of events with $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(A_n \text{ occur i.o.}) = 0$

- If $\{A_n : n \geq 1\}$ is a sequence of (pairwise) independent events & $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ then $\mathbb{P}(A_n \text{ occurs i.o.}) = 1$.

- $\mathbb{R} \ni m(X)$ is called a median of X if

$$\mathbb{P}(X \leq m(X)) \geq \tfrac{1}{2} \leq \mathbb{P}(X \geq m(X))$$

observe: $\varepsilon < \frac{1}{2}$ and $c > 0$ st.

$$\mathbb{P}(|X| > c) \leq \varepsilon < \tfrac{1}{2} \Rightarrow |m(X)| \leq c$$

17 Convergence in Probability, Almost sure Convergence and equivalence for sums of independent random variables

As we have seen in this course independence has been demonstrated as an important concept. Its a distinguishing feature of Probability theory vs Measure Theory.

- It is also crucial in applications and emphasised by Convergence of empirical mean (E.g. weak law of large numbers.)

- Also events involving independent σ-algebras have Special features (E.g Behavior of joint distribution function & Kolmogorov 0-1 law).

Proposition 1 (Equivalent formulations of $X_n \xrightarrow{P} X$)

(i) $\exists$ r.v. X such that $X_n \xrightarrow{P} X$ as $n \to \infty$

$\iff$ (ii) $\forall \varepsilon > 0$, $\sup_{m \geq n} \mathbb{P}(|X_m - X_n| > \varepsilon) \longrightarrow 0$ as $n \to \infty$

$\iff$ (iii) for every subsequence of $\{X_n : n \geq 1\}$ has a further subsequence Converging a.s. to the same r.v. X as in (i).

Proof:- (i) $\Rightarrow$ (ii) let $\varepsilon > 0$ be given

let $m \geq n$.

$$\{|X_m - X_n| > \varepsilon\} \subseteq \{|X_m - X| > \tfrac{\varepsilon}{2}\} \cup \{|X_n - X| > \tfrac{\varepsilon}{2}\}$$

$\Rightarrow \quad \mathbb{P}(|X_m - X_n| > \varepsilon) \le \mathbb{P}(|X_m - X| > \frac{\varepsilon}{2}) + \mathbb{P}(|X_n - X| > \frac{\varepsilon}{2})$ — (†)

From (i) $\exists N$: $m, n \ge N$

$$\mathbb{P}(|X_m - X| > \tfrac{\varepsilon}{2}) < \varepsilon \quad \text{— (‡)}$$
$$\mathbb{P}(|X_n - X| > \tfrac{\varepsilon}{2}) < \varepsilon$$

(†) and (‡) $\Rightarrow$ (ii). $\square$

(ii) $\Rightarrow$ (iii)

WLOG let us denote subsequence of $\{X_n : n \ge 1\}$ by $\{X_n\}_{n\ge 1}$ it self.

let $k \ge 1$ be given $\exists\, m_k \ge 1$ s.t.

$$\mathbb{P}(|X_n - X_m| > \tfrac{1}{2^k}) < \tfrac{1}{2^k} \quad \forall\ n \ge m \ge m_k \quad \text{— (1)}$$

let $n_1 = m_1$

$n_i = \max\{n_{i-1}, m_{i-1}\} \quad \forall i \ge 2$

$$A_i = \{|X_{n_i} - X_{n_{i+1}}| > \tfrac{1}{2^i}\}$$

By (1) $\quad \sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$

$\Rightarrow \quad A = \{\exists k_0 : |X_{n_{k+1}} - X_{n_k}| < \frac{1}{2^k}\ \forall k \ge k_0\}$

has $\quad \mathbb{P}(A) = 1$

Now $\quad |X_{n_\ell} - X_{n_s}| \le \sum_{k=s}^{\ell} |X_{n_{k+1}} - X_{n_k}| \quad \forall s \le \ell$

$\omega \in A$ if $k_0(\omega) \le s \le l$

$\Rightarrow$

$$|X_{n_l} - X_{n_s}| \le \sum_{k=s}^{l} \frac{1}{2^k} \le \frac{1}{2^{s-1}}$$

$$1 \le \mathbb{P}(A) \le \mathbb{P}\left(\bigcap_{u=1}^{\infty} \bigcup_{k_0=1}^{\infty} \bigcap_{s,l \ge k_0} \left\{ |X_{n_l} - X_{n_s}| < \frac{1}{u} \right\} \right)$$

$\Rightarrow$ $\{X_{n_k}\}_{k \ge 1}$ is a Cauchy sequence w.p. 1

$\Rightarrow$ $\exists$ a r.v X such that $X_{n_k} \xrightarrow{a.s.} X$ as $k \to \infty$.

$\Rightarrow$ $X_{n_k} \xrightarrow{p} X$ as $k \to \infty$ — 2

Let $\varepsilon > 0$ be given.

$$\mathbb{P}(|X_n - X| > \varepsilon) = \mathbb{P}\left(|X_n - X_{n_k}| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(|X_{n_k} - X| > \frac{\varepsilon}{2}\right)$$

By (ii) $\exists$ k_1: $n, n_k > k_1$

$$\mathbb{P}\left(|X_n - X_{n_k}| > \frac{\varepsilon}{2}\right) < \varepsilon$$

& by 2 $\exists$ k_2: $n_k \ge k_2$

$$\mathbb{P}\left(|X_{n_k} - X| > \frac{\varepsilon}{2}\right) < \varepsilon$$

$\Rightarrow$ $n \ge \max\{k_1, k_2\}$

$\Rightarrow$ $\mathbb{P}(|X_n - X| > \varepsilon) < \varepsilon$ (X does NOT depend on subsequence)

$\Rightarrow$ $X_n \xrightarrow{p} X$ as $n \to \infty$

As it $\exists \{n_k\}_{k\geq 1}$ and Y st.

$X_{n_{k_\ell}} \xrightarrow{a.s} Y$ $\Rightarrow$ $X_{n_{k_\ell}} \xrightarrow{p} Y$

but $X_{n_{k_\ell}} \xrightarrow{p} X$ from above

$\Rightarrow$ $X = Y$ (Ex.)

(iii) holds

(iii) $\Rightarrow$ (i)

Suppose $X_n \xrightarrow{p} X$ as $n \to \infty$ is not true

$\exists\ \varepsilon > 0$ and a subsequence X_{n_k} such that

$$P(|X_{n_k} - X| > \varepsilon_0) > \varepsilon_0 \quad \forall k \geq 1$$

there does not exists n_{k_ℓ} : $X_{n_{k_\ell}} \xrightarrow{p} X$ $\Rightarrow$ (iii) does not hold.

$\square$

Proposition 2 (Equivalent formulations of $X_n \xrightarrow{a.s} X$)

(i) $\exists$ r.v. X $X_n \xrightarrow{a.s.} X$ as $n \to \infty$

$\Leftrightarrow$

(ii) $\sup_{m \geq n} |X_m - X_n| \xrightarrow{p} 0$ as $n \to \infty$

Proof:- (i) $\Rightarrow$ (ii)

$X_n \xrightarrow{a.s.} X$ as $n \to \infty$

let $\varepsilon > 0$ be given.

$\Rightarrow$ (Ex.) $P(|X_n - X| > \varepsilon \text{ i.o.}) = 0$ —— (3)

$$A_n^\varepsilon = \sup_{m \geq n} |X_m - X| > \varepsilon = \bigcup_{m=n}^{\infty} |X_m - X| > \varepsilon.$$

and $$A_n^\varepsilon \downarrow \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} |X_m - X| \geq \varepsilon$$

$$= \{ |X_n - X| \geq \varepsilon \text{ i.o.} \}$$

$\therefore$ By 3

$$\lim_{n \to \infty} \mathbb{P}(A_n^\varepsilon) = 0$$

$$\Rightarrow \quad \mathbb{P}\left(\sup_{m \geq n} |X_m - X| > \varepsilon \right) \longrightarrow 0 \text{ as } n \to \infty \quad \text{--- } 4$$

Now: $$\mathbb{P}\left(\sup_{m \geq n} |X_m - X_n| > \varepsilon \right)$$

$$\leq \mathbb{P}\left(|X_n - X| > \frac{\varepsilon}{2} \right) + \mathbb{P}\left(\sup_{m \geq n} |X_m - X| > \frac{\varepsilon}{2} \right)$$

$\therefore$ As $X_n \xrightarrow{as} X$ as $n \to \infty$ $\Rightarrow$ $X_n \xrightarrow{p} X$ as $n \to \infty$

We have using 4

$$\mathbb{P}\left(\sup_{m \geq n} |X_m - X_n| > \varepsilon \right) \longrightarrow 0 \text{ as } n \to \infty$$

$\varepsilon > 0$ was arbitrary $\quad \therefore \quad \sup_{m \geq n} |X_m - X_n| \xrightarrow{p} 0$ as $n \to \infty$.

(ii) $\Rightarrow$ (i)

(ii) $\overset{\text{Ex}}{\Rightarrow}$ Proposition 1 (ii)

$\Rightarrow$ Proposition 1 (i)

$\therefore \exists\ X$ r.v. st $X_n \xrightarrow{P} X$ as $n \to \infty$ — (5)

let $\varepsilon > 0$ be given

$$\Rightarrow \mathbb{P}\left(\sup_{m \geq n} |X_m - X| > \varepsilon \right) \leq \mathbb{P}\left(\sup_{m \geq n} |X_m - X_n| > \frac{\varepsilon}{2} \right) + \mathbb{P}\left(|X_n - X| > \frac{\varepsilon}{2} \right)$$

by (5) and (ii) $\longrightarrow 0$ as $n \to \infty$. — (6)

Now,

$$\mathbb{P}(|X_n - X| > \varepsilon \quad \text{i.o.})$$
$$= \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} |X_m - X| > \varepsilon \right)$$
$$= \lim_{n \to \infty} \mathbb{P}\left(\bigcup_{m=n}^{\infty} |X_m - X| > \varepsilon \right)$$
$$= \lim_{n \to \infty} \mathbb{P}\left(\sup_{m \geq n} |X_m - X| > \varepsilon \right)$$

by (6) $= 0$ — (7)

Now $\mathbb{P}\left(\bigcap_{n=1} \bigcup_{m=1} |X_m - X| > \frac{1}{k} \right) = 0 \quad \forall\, k \geq 1$

by (7) with $\varepsilon = \frac{1}{k}$

$$\Rightarrow \mathbb{P}\left(\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=1} |X_m - X| > \frac{1}{k} \right) = 0$$

Ex

$\therefore \mathbb{P}(X_n \to X \text{ as } n \to \infty)$

$$= 1 - \mathbb{P}\left(\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} |X_m - X| > \frac{1}{k} \right) = 1$$

$\therefore X_n \xrightarrow{a.s.} X$ as $n \to \infty$ (i) holds $\square$

Lemma 1 (Levy Inequality) : If $\{Y_j : 1 \le j \le n\}$ are independent r.v.'s, $T_j = \sum_{n=1}^{j} Y_n$, then $\forall\, \varepsilon > 0$

ⓐ $$\mathbb{P}\left(\max_{1 \le j \le n} \left[T_j - m(T_j - T_n) \right] \geq \varepsilon \right) \le 2\, \mathbb{P}(T_n \geq \varepsilon)$$

ⓑ $$\mathbb{P}\left(\max_{1 \le j \le n} \left| T_j - m(T_j - T_n) \right| \geq \varepsilon \right) \le 2\, \mathbb{P}(|T_n| \geq \varepsilon)$$

Proof:- let $\varepsilon > 0$ be given.

$\boxed{\text{Same as } m(Y_{j+1} + \dots + Y_{n-j}) \in \mathbb{R}}$

Let

$$U = \begin{cases} \min\{ j : T_j - m(T_j - T_n) \geq \varepsilon \} & \text{if } \exists\, j.\ T_j - m(T_j - T_n) \geq \varepsilon \\ n+1 & \text{otherwise} \end{cases}$$

$$\{T_n \geq \varepsilon\} \supseteq \bigcup_{j=1}^{n} \{U = j\} \cap \{ T_n \geq T_j - m(T_j - T_n) \}$$

$$\therefore\ \mathbb{P}(T_n \geq \varepsilon) \geq \sum_{j=1}^{n} \mathbb{P}(U = j \cap \{ T_n \geq T_j - m(T_j - T_n) \})$$

(disjoint)

$\{U = j\} \in \sigma(Y_1, \dots, Y_j)$

$$\geq \sum_{j=1}^{n} \mathbb{P}(U = j)\, \mathbb{P}(T_n \geq T_j - m(T_j - T_n)).$$

$\{T_j - T_n \le m(T_j - T_n)\} \in \sigma(\{Y_{j+1}, \dots, Y_n\})$

$$= \sum_{j=1}^{n} \mathbb{P}(U = j)\, \mathbb{P}(T_j - T_n \le m(T_j - T_n))$$

$$\geq \mathbb{P}(1 \le U \le n)\ \frac{1}{2}$$

$$= \mathbb{P}\left(\max_{1 \le j \le n} \left[T_j - m(T_j - T_n) \right] \ge \varepsilon \right) \quad -(1)$$

$\Rightarrow$ (a)

In the proof of (a): let $\tilde{Y}_j = -Y_j$

and $\tilde{T}_j = \sum_{j=1}^{n} \tilde{Y}_j \qquad 1 \le j \le n$

Note: $m(\tilde{Y}_j) = m(-Y_j) = -m(Y_j)$

Using (a) for $\tilde{Y}_j$ we have

$$\mathbb{P}\left(\max_{1 \le j \le n} \left[\tilde{T}_j - m(\tilde{T}_j - \tilde{T}_n) \right] \ge \varepsilon \right) \le 2\,\mathbb{P}(\tilde{T}_n \ge \varepsilon) \quad -(2)$$

(2) + (1) $\Rightarrow$ (b) $\qquad \square$

Theorem 1 (Lévy's Theorem) let $\{X_n : n \ge 1\}$ be a sequence of independent random variables. If $S_n = \sum_{k=1}^{n} X_k$

then S_n converges a.s. $\iff$ S_n converges in Probability.

Proof:- $\Rightarrow$ is immediate from earlier result.

$\Leftarrow$ Let S_n converge to S in probability as $n \to \infty$. $\quad -(3)$

let $0 < \varepsilon < \frac{1}{2}$ be given. For $n \ge, m \ge 1$

$$\mathbb{P}(|S_n - S_m| > \varepsilon) \leq \mathbb{P}(|S_n - S| > \tfrac{\varepsilon}{2}) + \mathbb{P}(|S_m - S| > \tfrac{\varepsilon}{2})$$

. From 3 $\exists N$:

$$\left.\begin{aligned} &\mathbb{P}(|S_n - S_m| > \varepsilon) < \varepsilon \quad \forall\, n \geq m \geq N \\ \underset{\text{(observation)}}{\Rightarrow} \quad &m(S_n - S_m) \leq \varepsilon \quad \forall\, n \geq m \geq N \end{aligned}\right\} - 4$$

$\therefore$ Using 4, $k \geq m \geq N$

$$\mathbb{P}(\max_{m < n \leq k} |S_n - S_m| > 2\varepsilon)$$

$$\underset{4}{=} \mathbb{P}(\max_{m < n \leq k} |S_n - S_m| > 2\varepsilon, \max_{m < n \leq k} |m(S_k - S_n)| \leq \varepsilon)$$

$$\leq \mathbb{P}(\max_{m \leq n \leq k} |S_n - S_m - m(S_k - S_n)| > \varepsilon)$$

lemma $\underset{\text{inequality}}{\overset{\text{Levy}}{\leq}} 2\,\mathbb{P}(|S_k - S_m| > \varepsilon)$

$$\underset{4}{\leq} 2\varepsilon$$

$$\Rightarrow \quad \mathbb{P}(\sup_{m \leq n} |S_n - S_m| > 2\varepsilon) \leq 2\varepsilon$$

Proposition 1

$\Rightarrow$ $\exists$ a random variable S such that

$$S_n \xrightarrow{a.s} S \quad \text{as } n \to \infty \quad \square$$