Recall Conditional distributions of a random variable X

X - r.v. on Sample space S : $\varphi(B) = \mathbb{P}(X \in B \mid A)$

$A \subseteq S$ - event with $\mathbb{P}(A) > 0$

φ - called conditional distribution of X given the event A.

Example 3.2.6 :- $X \sim \text{uniform}(\{1,2\})$

Y - # of heads in X tosses of a fair coin } $Y = ?$

- $\mathbb{P}(Y=0 \mid X=1) = \frac{1}{2}$ & $\mathbb{P}(Y=1 \mid X=1) = \frac{1}{2}$

$Y \mid X=1 \sim \text{Bernoulli}(\frac{1}{2})$

- $\underbrace{\mathbb{P}(Y=0 \mid X=2) = \frac{1}{4}, \ \mathbb{P}(Y=1 \mid X=2) = \frac{1}{2} \ \& \ \mathbb{P}(Y=2 \mid X=2) = \frac{1}{4}}_{Y \mid X=2 \text{ - Conditional distribution.}}$

- Joint Distribution of X and Y

X and Y are discrete random variables, the joint distribution of X and Y is the probability

$$\varphi(a,b) = \mathbb{P}(X=a, Y=b)$$

where $a \in \text{Range}(X)$ and $b \in \text{Range}(Y)$

[Example Contd. 3.2.6] $\mathbb{P}(X=1, Y=0) = \mathbb{P}(Y=0 \mid X=1)\,\mathbb{P}(X=1) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

Similarly : $\mathbb{P}(X=a, Y=b)$ $a \in \{1,2\}$, $b \in \{0,1,2\}$.
one can compute

$p(X=x, Y=y)$	$X=1$	$X=2$
$Y=0$	$1/4$	$1/8$
$Y=1$	$1/4$	$1/4$
$Y=2$	0	$1/8$

Joint distribution is specified by the above table.

Joint distribution - conveys all information

Example 3.2.6. Contd. : $Y \in \{0,1,2\}$ $X \in \{1,2\}$

Already know: $Y \mid X=1$, $Y \mid X=2$ ✓

$\mathbb{P}(X=x, Y=y)$ ... ✓

i.e. $\mathbb{P}(X=1 \mid Y=0) = \mathbb{P}($ choosing 1 given 0 heads appeared $)$

$$= \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(Y=0)}$$

$$= \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(Y=0, X=1) + \mathbb{P}(Y=0, X=2)} = \frac{1/4}{1/4 + 1/8} = \frac{2}{3}$$

Q: Are X and Y independent? Distribution of X? Y?

Joint distribution :-

$p(X=x, Y=y)$	$X=1$	$X=2$	
$Y=0$	$1/4$	$1/8$	$3/8$ — $\mathbb{P}(Y=0)$
$Y=1$	$1/4$	$1/4$	$1/2$ — $\mathbb{P}(Y=1)$
$Y=2$	0	$1/8$	$1/8$ — $\mathbb{P}(Y=2)$
	$1/2 \equiv p(X=1)$	$1/2 \equiv P(X=2)$	

$$\underbrace{\mathbb{P}(X=1, Y=0) + \mathbb{P}(X=1, Y=1) + \mathbb{P}(X=1, Y=2)}_{\mathbb{P}\left(X=1 \cap \bigcup_{i=0}^{2} Y=i\right) \;=\; \mathbb{P}(X=1)}$$

Independence :- $\mathbb{P}(X=1, Y=0) = 1/4$ (table)

$$\mathbb{P}(X=1)\,\mathbb{P}(Y=0) = \underbrace{1/2}_{1^{st}\text{ column sum}} \cdot \underbrace{3/8}_{1^{st}\text{ row sum}} = 3/16$$

$3/16 \neq 1/4$ $\therefore$ X and Y are independent.

3.2.3 Memoryless Property of Geometric Distribution :-

$X \sim \text{Geometric}(1/2)$

$\equiv$ # of trial at which first Head appears in the tosses of a fair coin.

$$\underbrace{\mathbb{P}(X=k)}_{\text{Probability}} = (1-p)^{k-1}\, p \qquad \underbrace{k = 1,2,3,\ldots}_{\text{Range}(X)}, \quad p = \tfrac{1}{2}$$

Distribution of X

$$\mathbb{P}(X > m) = \mathbb{P}\left(\bigcup_{k=m+1}^{\infty} X = k\right)$$

Axiom 2

$$= \sum_{k=m+1}^{\infty} \mathbb{P}(X=k) = \sum_{k=m+1}^{\infty} \left(\tfrac{1}{2}\right)^{k-1} \tfrac{1}{2}$$

Ex:- Series Computation

$$= \sum_{k=m+1}^{\infty} \left(\tfrac{1}{2}\right)^{k} = \frac{1}{2^m}$$

Geometric Series

Q:- Given :- There are no heads in 1^{st} m-tosses. , what is the Probability of no heads in next n tosses ?

m tosses   m+n tosses

$$\mathbb{P}(X > m+n \mid X > m) = ?$$

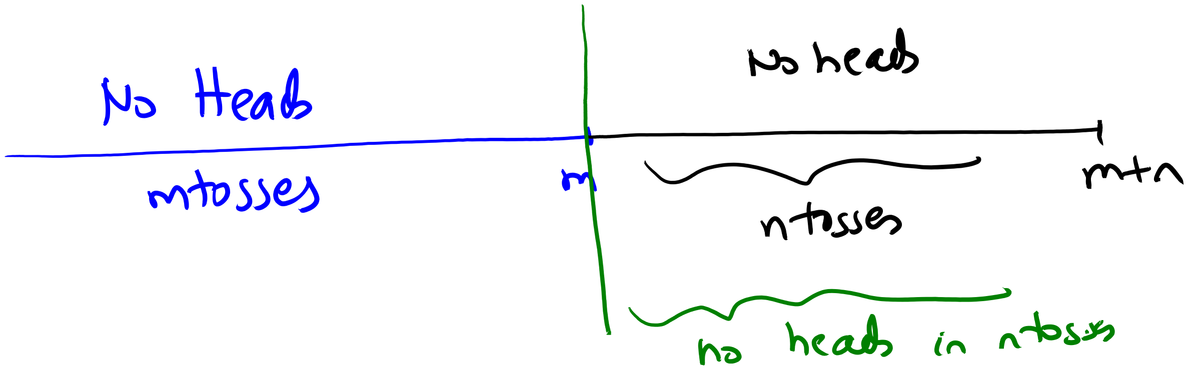
$$= \frac{\mathbb{P}(X > m+n,\ X > m)}{\mathbb{P}(X > m)}$$

$$= \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > m)} = \frac{\frac{1}{2^{m+n}}}{1/2^m}$$

$$= 1/2^n$$

let $m \geq 1$, $n \geq 1$

$$\mathbb{P}(X > m+n \mid X > m) = \frac{1}{2^n} = \mathbb{P}(X > n)$$



Memory less Property

$$\mathbb{P}(X > m+n \mid X > m) = \frac{1}{2^n} = \mathbb{P}(\text{no Heads in } n \text{ tosses})$$

$$\underbrace{X \mid (X > m)}_{\text{Range} = \{m+1,\ m+2, \dots\}} \sim \text{"Geometric}\left(\tfrac{1}{2}\right)\text{"}$$

" ↑ "

As $\mathbb{P}(X = m+k \mid X > m) = \mathbb{P}(X = k)$

3.2.4 Multinomial Distribution:

Example 3.2.12 :- Suppose an experiment has k outcomes

- each outcome j, has a probability p_j of occurring

- Perform n-trials (independent) of the above experiment

$X_j \equiv$ # of the n trials that result in j^{th} outcome.

observe :- $X_1 + X_2 + \cdots + X_k = n$ [not independent]

Q:- Find joint distribution of $(X_1, \ldots, X_n)$.

$$\mathbb{P}(X_1 = x_1, \ldots, X_k = x_k) = ?$$

$x_j \in \{0, 1, \ldots, n\}$

$\sum_{j=1}^{k} x_j = n$

$$B(x_1, \ldots, x_k) = \{X = x_1, \ldots, X_k = x_k\}$$

$$\mathbb{P}(B(x_1, \ldots, x_k)) = \sum_{\omega \in B(x_1 \ldots x_k)} \mathbb{P}(\{\omega\})$$

$\omega \in B(x_1, \ldots, x_k)$

Observe: in $\omega \equiv$ n tuple where outcome j appears x_j times

$p_j =$ Probability of outcome j

$$\therefore \quad \mathbb{P}(\{\omega\}) = \prod_{j=1}^{k} (p_j)^{x_j}$$

$$\Rightarrow \quad \mathbb{P}(B(x_1,\dots,x_k)) = \sum_{\omega \in B(x_1 \dots x_k)} \prod_{j=1}^{k} (p_j)^{x_j}$$

$$= |B(x_1 \dots x_k)| \prod_{j=1}^{k} (p_j)^{x_j}$$

observe

$|B(x_1,\dots,x_k)| \equiv$ # of ways of allocating n balls in k boxes : x_j falls in box j. $= \dfrac{n!}{x_1! \dots x_j!}$

$$\mathbb{P}(X_1 = x_1, \dots, X_k = x_k) = \begin{cases} \dfrac{n!}{x_1! \dots x_j!} \displaystyle\prod_{j=1}^{k} (p_j)^{x_j} & ; \quad x_j \in \{0,1,\dots,n\} \ \& \ \displaystyle\sum_{j=1}^{k} x_j = n \\ 0 & \text{otherwise} \end{cases}$$

$\square$

$\underline{K=2}$, ... connection to Binomial(n, p_1) or Binomial$(n, 1-p_1)$ — Ex

3.3 Functions of Random Variables

i.e. $\mathbb{P}(X=k)=1/5$
$k\in\{-2,-1,0,1,2\}$

Example 3.3.1. $X \sim \text{Uniform}(\{-2,-1,0,1,2\})$,

$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

$Y = f(X)$ Q: Distribution of Y?

(*'). $\text{Range}(Y) = \text{Range}\{f(x)\} = \{0, 1, 4\}$

$\mathbb{P}(Y=y) = ?$ $y \in \text{Range}(Y)$

$$
(*)\left\{
\begin{aligned}
\mathbb{P}(Y=0) &= \mathbb{P}(X=0) = 1/5 \\
\mathbb{P}(Y=1) &= \mathbb{P}(X=1 \cup X=-1) = \mathbb{P}(X=1) + \mathbb{P}(X=-1) \\
&= 1/5 + 1/5 = 2/5 \\
\mathbb{P}(Y=4) &= \mathbb{P}(X=2 \cup X=-2) = \cdots = 2/5
\end{aligned}
\right.
$$

(*') ; (*) $\equiv$ distribution of $f(X)$

Example 3.3.3. $X \sim \text{Bernoulli}(p)$ $Y \sim \text{Bernoulli}(p)$

$0 < p < 1$ & X and Y are independent

$Z = f(X,Y)$ & $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x+y$

Find Distribution of Z

Observe:

$$Z = X + Y \xrightarrow{\text{\# of success in 2 trials}} \stackrel{d}{=} \text{Binomial}(2,p)$$

$$\text{Range}(Z) = \{0,1,2\} \qquad \left(\because \begin{array}{l} \text{range}(X) = \{0,1\} \\ \text{range}(Y) = \{0,1\} \end{array}\right.$$

$$\mathbb{P}(Z=0) = \mathbb{P}(X=0, Y=0)$$

$$\stackrel{\text{independence}}{=} \mathbb{P}(X=0) \cdot \mathbb{P}(Y=0) = (1-p)(1-p)$$

$$= (1-p)^2$$

Similarly

$$\mathbb{P}(Z=1) = \mathbb{P}(X=0, Y=1 \cup X=1, Y=0)$$

$$= \mathbb{P}(X=0, Y=1) + \mathbb{P}(X=1, Y=0)$$

$$\stackrel{\text{independence}}{=} \mathbb{P}(X=0)\,\mathbb{P}(Y=1) + \mathbb{P}(X=1)\,\mathbb{P}(Y=0)$$

$$= (1-p)p + p(1-p) = 2p(1-p)$$

$$\mathbb{P}(Z=2) = \mathbb{P}(X=1, Y=1) = \mathbb{P}(X=1)\,\mathbb{P}(Y=1)$$

$$= p^2. \qquad \square$$