Recall :- • S - sample Space , $\mathcal{F}$ - Events , $\mathbb{P}$ - Probability

• Experiment :- all outcomes from the experiment

• $X : S \to \mathbb{R}$ is a random variable , S - Countable / finite

$\Rightarrow$ Range(X) - Countable - Discrete Random variables

Distribution of X

• $\underline{\text{Range}(X) \subseteq \mathbb{R}}$, $x \in \text{Range}(X)$ — $\mathbb{P}(X=x) := f_X(x)$

$\underline{f_X : \text{Range}(X) \to [0,1]}$ - Probability mass function of X.

Question :- [Bernoulli trials] "On average how many successes will
Example 2.1.2
be there after n trials ?"

4 - Summarising Discrete Random Variables :

Experiment: Roll a die : The outcomes are $\{1, 2, .., 6\}$ Average value of outcome?

Conventional idea: $\dfrac{1+2+3+4+5+6}{6} = 3.5$

$$\frac{1+2+3+4+5+6}{6} \equiv 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6)$$

weight each outcome according to the Probability of its outcome

Definition 4.1.1. Let $X: S \to T$ be a discrete random variable.
(i.e. S - countable / finite $\Rightarrow T$ is countable / finite)

Then the expected value of X is written as $E[X]$ and is given by

$$E[X] := \sum_{t \in T} t\, \mathbb{P}(X = t)$$ [Series Sum]

provided that the sum converges absolutely. In this case, we say X has "finite Expectation". If the sum diverges to $\pm\infty$, then we say X has infinite expectation. If the sum diverges but not $\pm\infty$, we say that the expected value of X is undefined.

Example 4.1.2 X - takes 3 values 200, 20, 0

Such that $\mathbb{P}(X = 200) = \frac{1}{1000}$, $\mathbb{P}(X = 20) = \frac{27}{1000}$

$$\mathbb{P}(X = 0) = \frac{972}{1000}$$

$$E[X] := \sum_{t \in T} t\, P(X = t) = 0 \cdot \frac{972}{1000} + 20 \cdot \frac{27}{1000} + 200 \cdot \frac{1}{1000}$$

$$= 0.74.$$

Theorem 4.1.3 $X : S \to T$ such that $X(s) = c$ for some $c \in \mathbb{R}$ $\forall s \in S$.

Then $E[X] = c$.

Proof:- $\mathbb{P}(X = c) = 1$

$$\Rightarrow \quad E[X] = c \cdot \mathbb{P}(X = c) = c \cdot 1 = c$$

$\square$

- This is written in short as: $E[c] = c$.

- $T = \text{Range}(X)$ is finite then $E[X] = \sum_{t \in T} t \, P(X = t) < \infty$,

and X has finite Expectation.

Example 4.1.4:- $\text{Range}(X) = \{2, 4, 8, 16, \ldots\}$

$$\mathbb{P}(X = 2^k) = \frac{1}{2^k} \quad \forall k \geq 1$$

$$E[X] = \sum_{t \in T} t \, \mathbb{P}(X = t) = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} \equiv \infty$$

$$\left[S_n = \sum_{k=1}^{n} 2^k \cdot \frac{1}{2^k} = \sum_{k=1}^{n} 1 = n \quad \Rightarrow \quad S_n \to \infty \text{ as } n \to \infty \right]$$

X has infinite Expectation.

Example 4.1.5

$$\text{Range}(X) = \{-2, 4, -8, 16, \ldots\}$$

$$\mathbb{P}(X = (-2)^k) = \frac{1}{2^k} \qquad k \geq 1$$

$$E[X] := \sum_{t \in T} t\, P(X=t) = \text{"} \sum_{k=1}^{\infty} (-2)^k \frac{1}{2^k} \text{"}$$

$$\left[S_n = \sum_{k=1}^{n} (-2)^k \frac{1}{2^k} = \sum_{k=1}^{n} (-1)^k = \begin{cases} -1 & n\text{-odd} \\ 0 & n\text{-even} \end{cases} \Rightarrow S_n \text{ - diverges [not to } \pm\infty] \right]$$

$E[X]$ is not defined.

Lemma 4.1.6: Let $X: S \to T$ be a discrete random variable. $E[X]$ is a real number if and only if $\underbrace{E[|X|] < \infty}$.

$(Y = |X| \;, \; E[Y])$

Proof to be done soon

Explanation / Significance: $X: S \to T$ is a random variable

Suppose $E[X] = \sum_{t \in T} t\, P(X=t) \equiv a \quad (a \in \mathbb{R})$

What does it tell you about X?

Proof of lemma 4.1.6 :

$$\text{Range}(X) = T$$
$$\text{Range}(|X|) = \{|t| : t \in T\} = \mathcal{U}$$

$$E[|X|] = \sum_{u \in \mathcal{U}} u \, \mathbb{P}(|X| = u)$$

$$E[X] = \sum_{t \in T} t \, \mathbb{P}(X = t)$$

By definition

- $\hat{T} = \{ t \in \mathbb{R} \mid |t| \in \mathcal{U} \}$ [as $u \in \mathcal{U}$ came from some $t \in T$ $\Rightarrow$ $\hat{T}$ contains T]

$t \in \hat{T}$ and $t \notin T$ $\Rightarrow$ $t \equiv$ is not in Range(X)

$\Rightarrow \mathbb{P}(X = t) = 0$.

$$E[X] = \sum_{t \in \hat{T}} t \, \mathbb{P}(X = t) \quad - (*)$$

observe: [$u \in \hat{T}$ and $-u \in \hat{T}$]

- $\underline{u \in \mathcal{U}}$ $\quad (|X| = u) = (X = u) \cup (X = -u)$

$$u \cdot \mathbb{P}(|X| = u) = u \left[\mathbb{P}(X = u) + \mathbb{P}(X = -u) \right]$$
$$= u \, \mathbb{P}(X = u) + u \, \mathbb{P}(X = -u) \quad - (**)$$
$$= |u| \, \mathbb{P}(X = u) + |-u| \, \mathbb{P}(X = -u)$$

[True even if $u = 0$]

$$\left.\begin{aligned}\sum_{u\in\mathcal{U}} u\cdot\mathbb{P}(|X|=u) &\overset{(\times\times)}{=} \sum_{u\in\mathcal{U}} |u|\,\mathbb{P}(X=u) + |-u|\,\mathbb{P}(X=-u)\\ &= \sum_{t\in\hat{T}} |t|\,\mathbb{P}(X=t)\end{aligned}\right\}\ (\times\times\times)$$

$$\overset{(\times)}{=} \sum_{t\in T} |t\cdot\mathbb{P}(X=t)|$$

$$\begin{aligned} E[X]<\infty &\iff \sum_{t\in T} t\,\mathbb{P}(X=t) \text{ converges absolutely}\\ &\iff \sum_{t\in T} |t\,\mathbb{P}(X=t)| < \infty\\ &\overset{(\times\times\times)}{\iff} \sum_{u\in\mathcal{U}} u\,\mathbb{P}(|X|=u) < \infty\\ &\iff E|X| < \infty \end{aligned}$$

$\square$

Theorem 4.1.7: Suppose X and Y are discrete random variables, both with finite expected value and defined on sample space S. Let $a,b\in\mathbb{R}$.

1. $E[aX] \overset{(*)}{=} aE[X]$
2. $E[X+Y] \overset{(*)}{=} E[X]+E[Y]$
3. $X\geq 0$ then $E[X]\geq 0$

(*) both sides are well defined

Proof :- ① To show $E[aX] = aE[X]$

$E[X] < \infty$ by assumption.

- $a = 0 \Rightarrow aX = 0 \Rightarrow E[0] = 0$

$\Rightarrow E[aX] = 0 = 0 \cdot E[X]$

$= aE[X]$

- $a \neq 0$ $\quad X : S \to \mathcal{U}$ $\quad \mathcal{U} = \text{range}(X)$

$Y = aX$ $\quad Y : S \to T$ $\quad T = \{au \mid u \in \text{Range}(X)\}$

$$E[aX] := E[Y] := \sum_{t \in T} t\,\mathbb{P}(Y = t) = \sum_{u \in \mathcal{U}} au\,\mathbb{P}(Y = au)$$

$$= \sum_{u \in \mathcal{U}} au\,\mathbb{P}(aX = au)$$

$$= a \sum_{u \in \mathcal{U}} u\,\mathbb{P}(X = u) = aE[X]$$

② To show $\underbrace{E[X+Y]}_{} = E[X] + E[Y]$

$$\left[Z = X + Y, \quad E[X+Y] \equiv E[Z] = \sum_{z \in \text{Range}(Z)} z\,\mathbb{P}(Z = z) \right]$$

let $Z = X + Y$ $\quad \text{Range}(Z) = \{u + v \mid u \in \text{Range}(X),\ v \in \text{Range}(Y)\}$

$$\therefore E[X+Y] \equiv E[Z] = \sum_{z \in \text{Range}(Z)} z\,\mathbb{P}(Z = z)$$

$$= \sum_{\substack{u \in \text{Range}(X) \\ v \in \text{Range}(Y)}} (u+v)\, \mathbb{P}(X=u, Y=v)$$

Rearrangement is okay as Series Converges absolutely

$$\overset{\circ}{=} \sum_{u \in \text{Range}(X)} \sum_{v \in \text{Range}(Y)} (u+v)\, \mathbb{P}(X=u, Y=v)$$

$$= \sum_{u \in \text{Range}(X)} u \sum_{v \in \text{Range}(Y)} \mathbb{P}(X=u, Y=v)$$

$$+ \sum_{v \in \text{Range}(Y)} v \sum_{u \in \text{Range}(X)} \mathbb{P}(X=u, Y=v)$$

$$\overset{\rightarrow}{=} \sum_{u \in \text{Range}(X)} u\, \mathbb{P}(X=u) + \sum_{v \in \text{Range}(Y)} v\, \mathbb{P}(Y=v)$$

$$\left[\rightarrow (X=u) = \bigcup_{v \in \text{Range}(Y)} (X=u, Y=v) \text{ and } (Y=v) = \bigcup_{u \in \text{Range}(X)} (X=u, Y=v) \right]$$

$$= E[X] + E[Y] \qquad \square$$