Recall: X, Y independent random variables with p.d.f $f_X(\cdot)$ and $f_Y(\cdot)$ respectively

[Sum] $Z = X+Y$ has a p.d.f given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

Example: $X \sim Exp(\lambda)$, $Y \sim Exp(\lambda)$

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z} & z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Z \sim Gamma(2, \lambda)$$

$W \sim Gamma(n, \lambda)$ if

$$f_W(w) = \begin{cases} \frac{\lambda^{n-1}}{(n-1)!} w^{n-1} e^{-\lambda w} & w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Ex:- $\left[W = X_1 + \dots + X_n \quad X_i \sim Exp(\lambda) \right]$

[Quotient] $Z = \frac{X}{Y}$; Z has p.d.f given by

$$f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_X(zy)\, f_Y(y)\, dy$$

Example 5.5.9 $X \sim$ Normal $(0, 1)$ $Y \sim$ Normal $(0,1)$

independent random variables.

$$Z = \frac{X}{Y}$$

$y \in \mathbb{R}$

$z \in \mathbb{R}$

$$f_X(zy)\, f_Y(y) = \frac{e^{-\frac{z^2y^2}{2}}}{\sqrt{2\pi}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{(1+z^2)}{2}y^2}}{2\pi}$$

$$f_Z(z) = \int_{-\infty}^{\infty} |y| \frac{e^{-\frac{(1+z^2)y^2}{2}}}{2\pi}\, dy$$

$$\overset{Ex}{=} \frac{1}{\pi} \int_0^\infty y\, e^{-\frac{(1+z^2)}{2}y^2}\, dy$$

$\left(u = (1+z^2)\frac{y}{2}\right.$, substitution$\left.\right)$ $$= \frac{1}{\pi(1+z^2)} \int_0^\infty e^{-t}\, dt$$

$$= \frac{1}{\pi(1+z^2)}$$

$Z \sim$ Cauchy (1).

6.1 Expectation and Variance

X = continuous random variable with p.d.f given by $f_X(\cdot)$

Def. 6.1.1:- $$E[X] = \underbrace{\int_{-\infty}^{\infty} x \, f_X(x) \, dx}$$ provided that the integral converges absolutely. X has "finite expectation".

If integral diverges to $\pm\infty$ then we say X has infinite expectation

If integral diverges not to $\pm\infty$ then we say $E[X]$ is not defined.

Example 6.1.2.: $X \sim \text{Uniform}(a,b)$ $a < b$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx = \int_a^b x \cdot \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b$$

$$= \frac{b+a}{2}$$

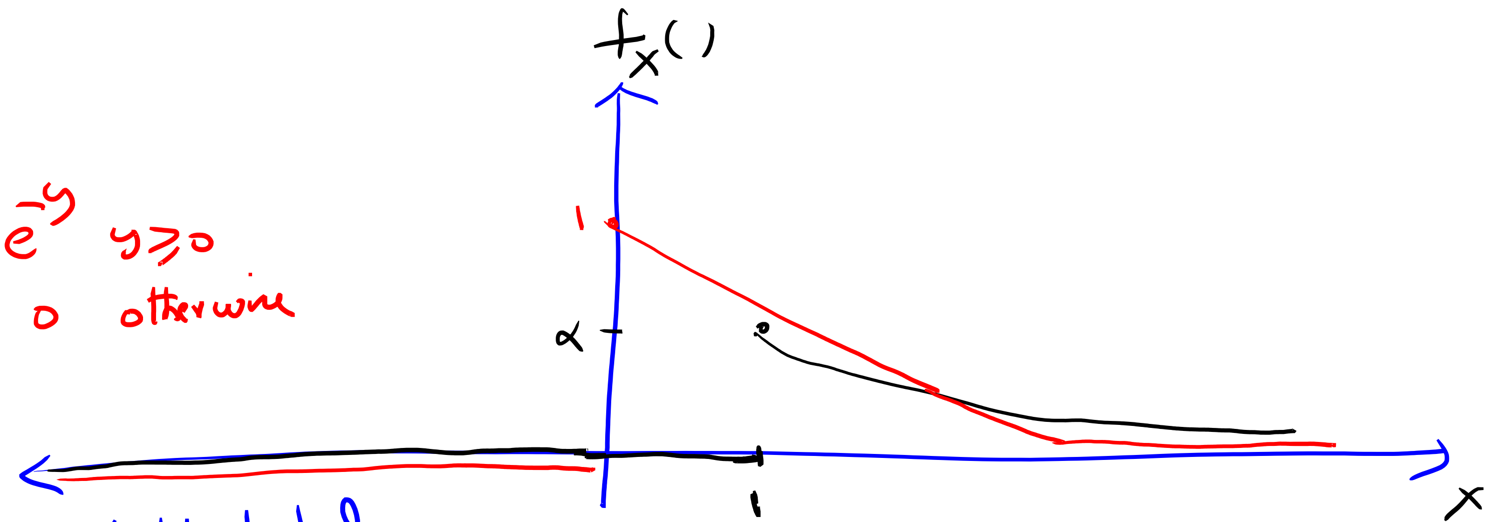
Notation
(Expected value $\leftrightarrow$ mean)

Example 6.1.3 [Pareto (α)] $\quad X \sim$ Pareto (α) if $\quad 0 < \alpha < 1$

$$f_X(x) = \begin{cases} \dfrac{\alpha}{x^{\alpha+1}} & 1 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$$

$Y \sim \operatorname{Exp}(1)$

$$f_Y(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$t \geq 1$ $\quad$ light tailed

$$\mathbb{P}(Y > t) = \int_t^\infty e^{-y}\,dy = e^{-t}$$

Tail is decaying exponentially

$$\mathbb{P}(X > t) = \int_t^\infty \frac{\alpha}{x^{\alpha+1}}\,dx = \frac{1}{t^\alpha}$$

Heavy tailed

Tail is decaying Polynomial

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_1^\infty \alpha x^{-\alpha}\,dx$$

$$= \alpha \lim_{M \to \infty} \int_1^M x^{-\alpha}\,dx$$

$$= \alpha \lim_{M \to \infty} \left[\frac{M^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1} \right]$$

for, $0 < \alpha < 1$ $\qquad = \infty$

Remark :- $X \sim \text{Cauchy}(1)$

$$E(X) \text{ "=" } \int_{-\infty}^{\infty} x \frac{1}{(1+x^2)\pi} \, dx \leftarrow$$ integral diverges to $\neq \pm\infty$

is not defined.

Theorem 6.1.5. X - random variable with p.d.f. $f_X(\,)$.

(a) $g : \mathbb{R} \to \mathbb{R}$ & $Z = g(X)$

$$E[Z] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx$$

(b) Y is another continuous random variable (X, Y) have a joint density $f(\,,\,)$

$h : \mathbb{R}^2 \to \mathbb{R}$ $\quad Z = h(X, Y)$

$$E[Z] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) f(x,y) \, dx \, dy$$

Proof :- OMIT. $\square$

- Use :- Don't need to find distribution of Z in (a) & (b) to compute $E[Z]$.

Example 6.1.7

$X, Y \sim$ Uniform $[0,1)$ - independent

$Z =$ "larger of X and Y" $= \max\{X, Y\}$

$E[Z] = ?$

Method 1:

- $F_Z(z) = \mathbb{P}(Z \le z)$

$= \mathbb{P}(\max(X,Y) \le z)$

$= \mathbb{P}(X \le z, Y \le z)$

(independent) $= \mathbb{P}(X \le z)\, \mathbb{P}(Y \le z)$

$$= \begin{cases} 0 & z < 0 \\ z^2 & 0 < z < 1 \\ 1 & 1 \le z \end{cases}$$

- $f_Z(z) = \begin{cases} 2z & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$

- $E[Z] = \int_{-\infty}^{\infty} z f_Z(z) = \int_0^1 z(2z)\, dz = 2\frac{z^3}{3}\Big|_0^1 = \frac{2}{3}$

Method 2 (use Theorem 6.1.5):

$$Z = \max(X, Y) := h(X,Y) \qquad h: \mathbb{R}^2 \to \mathbb{R}, \quad h(x,y) = \max\{x,y\}$$

$$E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dx\, dy$$

(Independence) $$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 \max\{x,y\}\; 1 \cdot 1 \; dx\, dy$$

$$\max\{x,y\} = \begin{cases} x & x \le y \\ y & y > x \end{cases}$$

$$= \int_0^1 \left[\int_0^y y\, dx + \int_y^1 x\, dx \right] dy$$

$$= \int_0^1 \left[yx\Big|_0^y + \frac{x^2}{2}\Big|_y^1 \right] dy$$

$$= \int_0^1 y^2\, dy + \frac{1}{2}\int_0^1 (1-y^2)\, dy$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \qquad \square$$

Theorem 6.1.8. X, Y are continuous random variables with joint p.d.f $f(\cdot,\cdot)$. Assume $E[X] < \infty$ and $E[Y] < \infty$.

let $a, b \in \mathbb{R}$

(a) $E[aX] = a E[X]$

(b) $E[aX+b] = a E[X] + b$

(c) $E[aX+bY] = a E[X] + b E[Y]$

(d) $X \geq 0$, $E[X] \geq 0$

Proof: Ex. $\square$

$X \sim$ continuous random variable with pdf $f_X(\cdot)$

$E[X] < \infty$

Definition 6.1.9:

$$Var[X] = E[(X - E[X])^2]$$

$$= \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\, dx$$

If integral converges $Var[X] < \infty$

or $Var[X] = \infty$

$$SD[X] = \sqrt{Var[X]}$$

Example 6.1.11 $X \sim \text{Normal}(0,1)$

$$f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad x \in \mathbb{R}$$

$$E[X] = \int_{-\infty}^{\infty} \frac{x\, e^{-x^2/2}}{\sqrt{2\pi}}\, dx = 0 \qquad [\text{Ex.}]$$

$(*)$ → integrand is odd & integral converges absolutely

$$\text{Var}[X] = E[(X - E[X])^2]$$

$$= E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x^2 e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx$$

$$\underset{\text{by parts}}{\overset{\text{integration}}{=}} 1 \qquad [\text{Ex}] \quad (**)$$

Theorem 6.1.10: X - has a p.d.f $f_X(\cdot)$, $E[X] < \infty$, $\text{Var}[X] < \infty$

(a) $\text{Var}[X] = E[X^2] - (E[X])^2$

(b) $\text{Var}[aX] = a^2 \text{Var}(X)$, $a \in \mathbb{R}$

(c) $\mathrm{var}[b + aX] = b + a^2 \mathrm{var}[X]$

· Y – has a p.d.f $f_Y(\cdot)$

(d) $E[XY] = E[X]\,E[Y]$ if Y is independent of X

(e) $\mathrm{var}[X+Y] = \mathrm{var}[X] + \mathrm{var}[Y] + 2\,\mathrm{Cov}[X,Y]$

Proof:

Ex.

$X \sim \mathrm{Normal}(\mu, \sigma^2)$ $\qquad f_X(x) = \dfrac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}, \; x \in \mathbb{R}$

$X = \sigma Z + \mu$ – (Seen before)

$Z \sim \mathrm{Normal}(0,1)$

[from above]

$E[X] = \sigma E[Z] + \mu$

$\overset{(*)}{=} \sigma \cdot 0 + \mu = \mu$

Theorems above

$\mathrm{Var}[X] = \sigma^2 \mathrm{Var}[Z] \overset{(**)}{=} \sigma^2 \cdot 1 = \sigma^2$

Markov Inequality : X : has p.d.f $f_X(\cdot)$, $c>0$

$$\mathbb{P}(X > c) = \int_c^\infty f_X(x)\, dx.$$

Suppose $f_X(x) = 0$, $x < 0$

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_0^\infty x\, f_X(x)\, dx$$

$$= \underbrace{\int_0^c x f_X(x) dx}_{\geq 0} + \int_c^\infty x f_X(x)\, dx$$

$$\geq \int_c^\infty x f_X(x)\, dx \qquad (\text{here } x > c)$$

$$\geq c \int_c^\infty f_X(x) dx$$

$$= c\, \mathbb{P}(X > c)$$

$$\Rightarrow \quad \mathbb{P}(X > c) \leq \frac{E[X]}{c} \qquad \text{if } f_X(x) = 0 \text{ when } x < 0$$

Similarly: $E[X] = \mu < \infty \quad Var[X] = \sigma^2 < \infty$

$$\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

[Proof is similar].

[Tschebyschev]

Basic Probability

M. Math

Random walk on graphs

Measure Theory

A

B

graph theory ⊗ not a prerequisite

Analysis / Probability

Martin Barlow
Grigoryan

Undergraduate Probability

Applications

Read:-

— Markov chains

—

Advance Probability — Available in an online Reading format

Videos + Q/A session